COEFFICIENT BOUNDS FOR CERTAIN CLASSES OF p-VALENT FUNCTIONS

LOH PART LEAM

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Alamat Tetap: SL 20 Kampung Coldstream, 35500 Bidor, Perak.

Tarikh: 2 Mac 2016

Disahkan oleh,

(Tandatangan Pustakawan)

(PROF. MADYA DR. AINI JANTENG)
Penyelia

DECLARATION

I hereby acknowledge that I have stated the source of each quotations, equations, summaries and reference in this study.

21 December 2015

Loh Part Leam

MS 1211 011 T

CERTIFICATION

NAME : **LOH PART LEAM**

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1. SUPERVISOR

Assoc. Prof. Dr. Aini Janteng

Signature

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ABSTRACT

This study considers U the class of functions which are analytic in the open unit disk $D = \{z: |z| < 1\}$ given by $w(z) = \sum_{n=1}^{\infty} b_n z^n$ and satisfying the conditions $w(0) = 0, |w(z)| < 1, z \in D$. The subclass of U consisting of univalent functions and normalized by the conditions $f(0) = f'(0) - 1 = 0$ is denoted by S. This study also considers $A(p)$ the class of functions defined by $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$, where a_{p+n} is complex number and $p \in \mathbb{N}$, which are analytic in the open unit disc D. The subclass of $A(p)$ denoted by $T(p)$, consisting of functions f of the form $f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$, where $a_{p+n} \geq 0$ and $p \in \mathbb{N}$. By considering functions $f \in T(p)$, a new subclass is proposed and coefficient estimates, growth and distortion theorem, closure theorem and extreme points are obtained for this class. In a meanwhile, the upper bounds for the Fekete-Szegö and second Hankel functional are obtained for certain subclasses of $A(p)$.

ABSTRAK

BATASAN PEKALI BAGI SUATU KELAS FUNGSI p-VALEN

Kajian ini mempertimbangkan U *sebagai kelas fungsi yang analisis dalam cakera unit terbuka* $D = \{z: |z| < 1\}$ *dengan* $w(z) = \sum_{n=1}^{\infty} b_n z^n$ *dan memenuhi syarat* $w(0) = 0, |w(z)| < 1, z \in D$. *Subkelas bagi* U *yang terdiri daripada fungsi univalen dan ternormal dengan syarat* $f(0) = f'(0) - 1 = 0$ *dilambangkan sebagai* S. *Kajian ini juga mempertimbangkan* $A(p)$ *sebagai kelas fungsi* $f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$, *dengan* a_{p+n} *adalah nombor kompleks dan* $p \in \mathbb{N}$, *yang analisis dalam cakera unit terbuka* D. *Subkelas* $A(p)$ *dilambangkan dengan* $T(p)$, *terdiri daripada fungsi* f *berbentuk* $f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$, *dengan* $a_{p+n} \geq 0$ *dan* $p \in \mathbb{N}$. *Dengan mempertimbangkan fungsi* $f \in T(p)$, *suatu subkelas diperkenalkan dan anggaran pekali, pertumbuhan dan teorem herotan, teorem tutupan serta titik ekstrim diperoleh bagi fungsi di dalam kelas ini. Di samping itu, batasan atas bagi fungsian Fekete-Szegö dan penentu Hankel ke-2 juga diperoleh bagi subkelas* $A(p)$.

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LIST OF SYMBOLS

$+$	-	Addition
$-$	-	Subtraction
$=$	-	Equal
$<$	-	Less than
$\leq$	-	Less than or equal to
$>$	-	Greater than
$\geq$	-	Greater than or equal to
$\prec$	-	Subordinate to
$\lvert\ \rvert$	-	Modulus
∞	-	Infinity
e	-	Exponential
$\max$	-	Maximum value
$\lim$	-	Limit
$\sum$	-	Summation
$\prod$	-	Product
$\in$	-	Element of
$\subset$	-	Subset of
Re	-	Real part of
$\mathbb{C}$	-	Complex plane
$\mathbb{N}$	-	Set of natural number
$\mathbb{R}$	-	Set of real number
C_R	-	Circle
D	-	Open unit disc
D_R	-	Closed disk
R	-	Radius
E	-	Domain
U	-	Class of analytic functions
S	-	Class of univalent functions
T	-	Class of functions with negative coefficients
P	-	Class of functions with positive real part
$A(p)$	-	Class of p-valent functions

$T(p)$	-	Class of p-valent functions with negative coefficients
S^*	-	Class of starlike functions
C	-	Class of convex functions
K	-	Class of close-to-convex functions
$S_s^*M(p,\alpha,\beta,\delta)$	-	Class of functions $f \in T(p)$ which satisfy $\left\lvert\frac{zf'(z)}{f(z)-f(-z)} - (p+\delta)\right\rvert < \beta\left\lvert\frac{\alpha zf'(z)}{f(z)-f(-z)} + (p-\delta)\right\rvert$
$S_c^*M(p,\alpha,\beta,\delta)$	-	Class of functions $f \in T(p)$ which satisfy $\left\lvert\frac{zf'(z)}{f(z)+\overline{f(\bar{z})}} - (p+\delta)\right\rvert < \beta\left\lvert\frac{\alpha zf'(z)}{f(z)+\overline{f(\bar{z})}} + (p-\delta)\right\rvert$
$S_{sc}^*M(p,\alpha,\beta,\delta)$	-	Class of functions $f \in T(p)$ which satisfy $\left\lvert\frac{zf'(z)}{f(z)-\overline{f(\overline{-z})}} - (p+\delta)\right\rvert < \beta\left\lvert\frac{\alpha zf'(z)}{f(z)-\overline{f(\overline{-z})}} + (p-\delta)\right\rvert$
$S_sT(p,\lambda,\alpha,\beta,\delta)$	-	Class of functions $f \in T(p)$ which satisfy $\left\lvert\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)[f(z)-f(-z)]+\lambda z[f(z)-f(-z)]'} - (p+\delta)\right\rvert < \beta\left\lvert\frac{\alpha[zf'(z)+\lambda z^2f''(z)]}{(1-\lambda)[f(z)-f(-z)]+\lambda z[f(z)-f(-z)]'} + (p-\delta)\right\rvert$
$M_s(\lambda)$	-	Class of functions $f \in S$ which satisfy $Re\left(\frac{zf'(z)+\lambda z^2f''(z)}{(1-\lambda)[f(z)-f(-z)]+\lambda z[f(z)-f(-z)]'}\right) > 0$
S_s^*	-	Class of starlike functions with respect to symmetric points
C_s	-	Class of convex functions with respect to symmetric points
$S^*(\alpha)$	-	Starlike function of order alpha
$C(\alpha)$	-	Convex function of order alpha
$S_s^*M(\alpha,\beta,\delta)$	-	Class of functions $f \in T$ which satisfy $\left\lvert\frac{zf'(z)}{f(z)-f(-z)} - (1+\delta)\right\rvert < \beta\left\lvert\frac{\alpha zf'(z)}{f(z)-f(-z)} + (1-\delta)\right\rvert$
$S_s^*T(\alpha,\beta)$	-	Class of functions $f \in T$ which satisfy $\left\lvert\frac{zf'(z)}{f(z)-f(-z)} - 1\right\rvert < \beta\left\lvert\frac{\alpha zf'(z)}{f(z)-f(-z)} + 1\right\rvert$
$C_sT(\alpha,\beta)$	-	Class of functions $f \in T$ which satisfy $\left\lvert\frac{(zf'(z))'}{(f(z)-f(-z))'} - 1\right\rvert < \beta\left\lvert\frac{\alpha(zf'(z))'}{(f(z)-f(-z))'} + 1\right\rvert$

$S_s^*(A,B)$ - Class of functions $f \in S$ which satisfy

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+Az}{1+Bz}, \qquad -1 \le B < A \le 1, z \in D.$$

$S_s^*(p,A,B)$ - Class of functions $f \in A(p)$ which satisfy

$$\frac{2zf'(z)}{p[f(z)-f(-z)]} \prec \frac{1+Az}{1+Bz}, \qquad -1 \le B < A \le 1, z \in D.$$

$C_s(A,B)$ - Class of functions $f \in S$ which satisfy

$$\frac{2[zf'(z)]'}{[f(z)-f(-z)]'} \prec \frac{1+Az}{1+Bz}, \qquad -1 \le B < A \le 1, z \in D.$$

$C_s(p,A,B)$ - Class of functions $f \in A(p)$ which satisfy

$$\frac{2[zf'(z)]'}{p[f(z)-f(-z)]'} \prec \frac{1+Az}{1+Bz}, \qquad -1 \le B < A \le 1, z \in D.$$

$M_s(p,b,\lambda)$ - Class of functions $f \in A(p)$ which satisfy

$$Re\left(1+\frac{1}{b}\left[\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)[f(z)-f(-z)]+\lambda z[f(z)-f(-z)]'}-1\right]\right) > 0$$

$M_s(p,b,\lambda,A,B)$ - Class of functions $f \in A(p)$ which satisfy

$$1+\frac{1}{b}\left[\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)[f(z)-f(-z)]+\lambda z[f(z)-f(-z)]'}-1\right] \prec \frac{1+Az}{1+Bz},$$

where $-1 \le A < B \le 1$ and $z \in D$.

$M_s(\lambda,A,B)$ - Class of functions $f \in S$ which satisfy

$$\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)[f(z)-f(-z)]+\lambda z[f(z)-f(-z)]'} \prec \frac{1+Az}{1+Bz},$$

where $-1 \le A < B \le 1$ and $z \in D$.

$S_p(b,\lambda,\alpha)$ - Class of functions $f \in A(p)$ which satisfy

$$Re\left(1+\frac{1}{b}\left(\frac{\lambda z^2 f''(z)+zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}-1\right)\right) < \alpha$$

$M_p(b,\alpha)$ - Class of functions $f \in A(p)$ which satisfy

$$Re\left(1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right) < \alpha$$

$N_p(b,\alpha)$ - Class of functions $f \in A(p)$ which satisfy

$$Re\left(1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right) < \alpha$$

$M_p(\alpha)$ - Class of functions $f \in A(p)$ which satisfy

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha$$

$N_p(\alpha)$	- Class of functions $f \in A(p)$ which satisfy $$Re\left(1+\frac{zf''(z)}{f'(z)}\right)<\alpha$$
$C(\lambda,\alpha)$	- Class of functions $f \in S$ which satisfy $$Re\left(\frac{\lambda z^3 f'''(z)+(2\lambda+1)z^2 f''(z)+zf'(z)}{\lambda z^2 f''(z)+zf'(z)}\right)>\alpha$$
$C_p(b,\lambda,\alpha)$	- Class of functions $f \in A(p)$ which satisfy $$Re\left(1+\frac{1}{b}\left(\frac{\lambda z^3 f'''(z)+(2\lambda+1)z^2 f''(z)+zf'(z)}{\lambda z^2 f''(z)+zf'(z)}-1\right)\right)<\alpha$$
$M_C(\lambda)$	- Class of functions $f \in S$ which satisfy $$Re\left(\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\lambda)(g(z)-g(-z))+\lambda z(g(z)-g(-z))'}\right)>0$$
$ST_p(\alpha)$	- Class of functions $f \in A(p)$ which satisfy $$Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha$$
$CV_p(\alpha)$	- Class of functions $f \in A(p)$ which satisfy $$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha$$
$M_p(\alpha)$	- Class of functions $f \in A(p)$ which satisfy $$Re\left((1-\alpha)\frac{zf'(z)}{pf(z)}+\alpha\frac{(zf'(z))'}{pf'(z)}\right)>0$$
$S^*(\varphi)$	- Class of functions which satisfy $$\frac{zf'(z)}{f(z)}\prec\varphi(z)$$
$C(\varphi)$	- Class of functions which satisfy $$1+\frac{zf''(z)}{f'(z)}\prec\varphi(z)$$
$S_S^*(p,\varphi)$	- Class of functions $f \in A(p)$ which satisfy $$\frac{2zf'(z)}{p[f(z)-f(-z)]}\prec\varphi(z)$$
$C_s(p,\varphi)$	- Class of functions $f \in A(p)$ which satisfy $$\frac{2[zf'(z)]'}{p[f(z)-f(-z)]'}\prec\varphi(z)$$

CHAPTER 1

PRELIMINARIES

1.1 Introduction

This thesis is about geometric function theory. According to (Bulboacă, Cho and Kanas, 2012), geometric function theory is the branch of complex analysis which deals with the geometric properties of analytic functions, founded around the turn of 20^{th} century. In spite of the famous coefficient problem, the Bieberbach conjecture that was solved by Louis de Branges in 1984 suggest various approaches and directions of studied in geometry function theory. According to (Ahuja, 1986), the study of geometric function theory is one of the most fascinating aspects of the theory of analytic functions of a complex variable. Furthermore, according to (Crowdy, 2008), geometric function theory is an area of mathematics characterized by an intriguing marriage between geometry and analysis.

1.2 Analytic and Univalent Functions

This section begins with the well-known definition about analytic functions. According to (Duren, 1983), a complex-valued function f of a complex variable is differentiable at a point $z_0 \in \mathbb{C}$ if it has a derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

at z_0. Such a functions f is analytic at z_0 if it is differentiable at every point in some neighbourhood of z_0. In (Sharma and Sharma, 2000), a function f is analytic at a point z_0, if it is defined and possesses a derivative at every point in some neighbourhood of z_0.

In this thesis, let U be the class of functions which are analytic in the open unit disk $D = \{z: |z| < 1\}$ given by

$$w(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.1}$$

and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$, b_n is a complex number and $z \in D$.

Next, we give the definition of univalent functions. According to (Goodman ,1975), a functions $f(z)$ is said to be univalent in a domain E if it provides a one-to-one mapping onto its image, $f(E)$. The following gives the definition of univalent functions.

Definition 1.1 (Goodman, 1975) A function $f(z)$ is said to be univalent in a domain E if the condition $f(z_1) = f(z_2)$, implies that $z_1 = z_2$, with $z_1, z_2 \in E$.

In (Duren, 1983) , a single-valued function f is said to be univalent (or schlicht) in a domain $E \subset \mathbb{C}$ if it never takes the same value twice that is , if $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in E with $z_1 \neq z_2$.

This thesis also considers the subclass of U consisting of univalent functions and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ which is denoted by S. Thus, each $f \in S$ has a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

where a_n is a complex number and $z \in D$.

The subclass of S, denoted by T consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \tag{1.3}$$

where $a_n \geq 0, a_n \in \mathbb{R}$

The class of T was introduced by (Silverman, 1975) and many researchers have studied the class T such as (Srivastava, Owa and Chatterjea, 1987), (Joshi and Srivastava, 1996), (Keerthi, Gangadharan and Srivastava, 2008), and (Jain, 2012).

1.3 Subclasses of S

The important subclasses of S include of class of starlike functions denoted as S^*, class of convex functions denoted as C and class of close-to-convex functions denoted as K. Hence, this section defines the main subclasses of S as follows.

Definition 1.2 (Goodman, 1975) A set E in the plane is said to be starlike with respect to w_0 an interior point of E if each ray with initial point w_0 intersects the interior of E in a set that is either a line segment or a ray. If a function $f(z)$ maps D onto a domain that is starlike with respect to w_0, then we say that $f(z)$ is starlike with respect to w_0. In the special case that $w_0 = 0$ we say that $f(z)$ is a starlike function.

Theorem 1.1 (Goodman, 1975) Let $f(z)$ be analytic and univalent in the closed disk $D_R: |z| \leq R < 1$. Then $f(z)$ maps D_R onto a region that is starlike with respect to $w = 0$ if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0$$

for z on circle $C_R: |z| < R$.

According to (Jenkins, 1958), the class of starlike functions first treated by Alexander in 1915 and later by Nevanlinna in 1922. Many researchers have studied the class S^* such as (Ghanim and Darus, 2010), (Hayami and Owa, 2010) and (Nishiwaki and Owa, 2013).

According to (Goodman, 1975), the Koebe function which is given by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$$

is a starlike function.

The class S^* can be generalized to the class of starlike functions of order α, $S^*(\alpha)$ and satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad 0 \leq \alpha < 1, \qquad z \in D.$$

The class $S^*(\alpha)$ was introduced by (Robertson, 1936).

Definition 1.3 (Goodman, 1975) A set E in the plane is called convex if for every pair of points w_1 and w_2 in the interior of E, the line segment joining w_1 and w_2 is also in the interior of E. If a function $f(z)$ maps D onto a convex domain, then $f(z)$ is called a convex functions.

Theorem 1.2 (Goodman, 1975) Let $f(z)$ be analytic and univalent in the closed disk $D_R: |z| \leq R < 1$. Then $f(z)$ maps D_R onto a convex domain if and only if

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0$$

for z on circle $C_R: |z| = R$.

According to (Jenkins, 1958), the first special subclass of S to be treated was that of convex functions introduced by Study in 1913. Many researchers have studied the class C such as (Xu, Gui and Srivatava, 2011), (Hayami and Owa, 2011) and (Ali, Nargesi and Ravichandran, 2013).

According to (Goodman, 1975), the Möbius function

$$L_0(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n \tag{1.4}$$

is a convex function because it maps D onto a half-plane.

The class C can be generalized to the class of convex functions of order α, $C(\alpha)$ and satisfying the condition

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, \;\; 0 \leq \alpha < 1, \;\; z \in D\,.$$

The class $C(\alpha)$ was introduced by (Robertson, 1936).

We now turn to an interesting subclass of S which contain S^* and has a simple geometry description. This is the class of close-to-convex functions which was introduced by (Kaplan, 1952). A functions $f(z)$ analytic in the unit disc is said to be close-to-convex if there is a convex $g(z)$ such that

$$Re\left(\frac{f'(z)}{g'(z)}\right) > 0\text{, for all } z \in D.$$

Every convex function is close-to-convex. More generally, every starlike function is close-to-convex. Indeed each $f \in S^*$ has the form $f(z) = zg'(z)$ for some $g \in C$ and

$$Re\left(\frac{f'(z)}{g'(z)}\right) = Re\left(\frac{zf'(z)}{f(z)}\right) > 0.$$

These remarks are summarized by the chain of properties inclusion $C \subset S^* \subset K$.

Many researchers have studied the class of close-to-convex function such as (El-Ashwah and Thomas, 1987), (Srivastava, Mishra and Das, 2001), (Gao and Zhou, 2005), (Mehrok, Singh and Gupta, 2011), (Mehrok and Singh, 2011b) and (Tang and Deng, 2013).

1.4 p-valent Functions

This section gives the definition of p-valent functions which is also known as multivalent functions. The theory of p-valent functions is the generalization of the theory of univalent function. The following is the definition of p-valent function.

Definition 1.4 (Goodman, 1975) A function $f(z)$ meromorphic in a domain E is said to be p-valent in E (or multivalent of order p in E) if for each w_0 (infinity included) the equation $f(z) = w_0$ has a most p root in E (where the root are counted in accordance with their multiplicity) and if there is some w_1 such that the equation $f(z) = w_1$ has exactly p root in E.

In this thesis, let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \tag{1.5}$$

which are analytic and p-valent in D and $p \in \mathbb{N} = \{1,2,3,\ldots\}$.

Next, let $T(p)$ denote the subclass of $A(p)$ consisting of functions which are analytic and p-valent which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \tag{1.6}$$

with $a_{p+n} \geq 0, p \in \mathbb{N} = \{1,2,3, \dots\}$.

Many researchers have studied of the class $T(p)$ such as (Goel and Sohi, 1981), (Owa and Obradovic, 1988), (Aouf, 1989) and (Amsheri and Zharkova, 2011).

1.5 Subordination Principle

This section gives the definition of subordination which will be used to define new subclasses of $A(p)$ in Chapters 3 and 4. The following is the definition of subordination:

Definition 1.5 (Owa, 1986) Let $f(z)$ and $g(z)$ be analytic in the open unit disk $D = \{z: |z| < 1\}$. A function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $\phi(z)$ analytic in the unit disc D satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$, $z \in D$ such that $f(z) = g(\phi(z))$ for $z \in D$. We denote by $f(z) \prec g(z)$ this relation. In particular, if $g(z)$ is univalent in the unit disc D the subordination is equivalent to $f(0) = g(0)$ and range of $f(z) \subset$ range of $g(z)$.

1.6 Function with Positive Real Part

Closely related to the classes S^* and C is the class of all functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{1.7}$$

be analytic in D and satisfies the condition $p(0) = 1, Re(p(z)) > 0$.

According to (Polatoğlu and Bolcal, 2000) this function is called Carathéodory functions. The class of these functions is denoted by P. The Möbius function which is given by (1.4) play a central role in the class P. This function is in the class P, it is analytic and univalent in D, and it maps D onto the half-plane (Goodman, 1975).

By using the subordination principle, we have

$$p(z) \in P \text{ if and only if } p(z) \prec \frac{1+z}{1-z}.$$

1.7 Objectives of Study

The objectives of this study are:

(i). to propose a new subclass of $T(p)$ and determine the properties of functions in this class which include coefficient estimates, growth and distortion theorem, closure theorem and extreme points;

(ii). to determine the upper bounds of the coefficients and Fekete-Szegö inequality for functions in certain subclasses of $A(p)$; and

(iii). to determine the upper bounds of the second Hankel determinant for functions in certain subclasses of $A(p)$.

1.8 Thesis Outline

This study consists of five chapters. It begins with Chapter 1 which gives an overview on the geometric function theory and some definitions of functions which will be referred throughout this study. Chapter 2 giving a new subclass of $T(p)$ which is denoted by $S_sT(p,\lambda,\alpha,\beta,\delta)$ and obtaining the coefficient estimates, growth and distortion theorem, closure theorem and extreme points of $f \in S_sT(p,\lambda,\alpha,\beta,\delta)$. Next, Chapter 3 giving the upper bounds of the the coefficients and Fekete-Szegö inequality for certain subclasses of $A(p)$ which are denoted by $S_s^*(p,A,B)$, $C_s(p,A,B)$, $M_s(p,b,\lambda,A,B)$, $S_p(b,\lambda,\alpha)$ and $C_p(b,\lambda,\alpha)$. Whereas, the upper bounds of the second Hankel determinant for certain subclasses of $A(p)$ which are denoted by $S_S^*(p,\varphi)$ and $C_s(p,\varphi)$ are discussed in Chapter 4. Finally, this study ends with conclusion and future works in Chapter 5.

CHAPTER 2

THE CLASS $S_sT(p,\lambda,\alpha,\beta,\delta)$

2.1 Introduction

This chapter develops new subclass of $T(p)$ which are denoted by $S_sT(p,\lambda,\alpha,\beta,\delta)$ with $p = 1,3,5,7,\ldots$, $0 \le \lambda \le 1$, $0 \le \alpha < 1$, $0 < \beta < 1$, $0 \le \delta < \frac{n-p}{2}$ and $n > p$. Some properties of functions in the class $S_sT(p,\lambda,\alpha,\beta,\delta)$ which include coefficient estimates, growth and distortion theorem, closure theorem and extreme points are obtained. The reason of developing $S_sT(p,\lambda,\alpha,\beta,\delta)$ and investigating the properties of functions belonging to this class were motivated by the original idea of (Khairnar and Rajas, 2010) and (Singh, 2013).

2.2 Class $S_sT(p,\lambda,\alpha,\beta,\delta)$

This section begins by giving the previous known definitions of classes which are introduced by (Khairnar and Rajas, 2010).

Definition 2.1 (Khairnar and Rajas, 2010) A function $f \in S_s^*M(p,\alpha,\beta,\delta)$ if it satisfies

$$\left|\frac{zf'(z)}{f(z)-f(-z)} - (p+\delta)\right| < \beta\left|\frac{\alpha zf'(z)}{f(z)-f(-z)} + (p-\delta)\right|$$

for $p \in \mathbb{N}$, $0 \le \alpha < 1$, $0 < \beta < 1$, $0 \le \delta < p$ and $z \in D$.

Definition 2.2 (Khairnar and Rajas, 2010) A function $f \in S_c^*M(p,\alpha,\beta,\delta)$ if it satisfies

$$\left|\frac{zf'(z)}{f(z)+\overline{f(\bar{z})}} - (p+\delta)\right| < \beta\left|\frac{\alpha zf'(z)}{f(z)+\overline{f(\bar{z})}} + (p-\delta)\right|$$

for $p \in \mathbb{N}$, $0 \le \alpha < 1$, $0 < \beta < 1$, $0 \le \delta < p$ and $z \in D$.

Definition 2.3 (Khairnar and Rajas, 2010) A function $f \in S_{sc}^{*}M(p,\alpha,\beta,\delta)$ if it satisfies

$$\left|\frac{zf'(z)}{f(z)-\overline{f(-\bar{z})}}-(p+\delta)\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)-\overline{f(-\bar{z})}}+(p-\delta)\right|$$

for $p \in \mathbb{N}$, $0 \le \alpha < 1$, $0 < \beta < 1$, $0 \le \delta < p$ and $z \in D$.

The above conditions imposed on α, β and δ are necessary to ensure that these classes form a subclass of $A(p)$.

Khainar and Rajas (2010) have obtained some of the basic properties for functions f belongs to the classes $S_{s}^{*}M(p,\alpha,\beta,\delta)$, $S_{c}^{*}M(p,\alpha,\beta,\delta)$ and $S_{sc}^{*}M(p,\alpha,\beta,\delta)$ such as growth and distortion theorems, closure theorems, extreme points and convolution theorems.

In 2013, Singh considered the class $M_s(\lambda)$ as defined below.

Definition 2.4 (Singh, 2013) Let $M_s(\lambda)$ be the subclass of functions $f(z) \in A$ and satisfying the condition

$$Re\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)[f(z)-f(-z)]+\lambda z[f(z)-f(-z)]'}\right)>0,$$

for $0 \le \lambda \le 1$, $z \in D$.

Class $M_s(\lambda)$ was subclass for $M_s(\lambda, A, B)$ that introduced by (Selvaraj and Vasanthi, 2011). Obviously $M_s(0) = S_s^{*}$, the class of starlike functions with respect to symmetric points introduced by (Sakaguchi, 1959) and $M_s(1) = C_s$, the class of convex functions with respect to symmetric points introduced by (Das and Singh, 1977). For a class $M_s(\lambda)$, (Singh, 2013) has obtained the sharp upper bounds of the second Hankel determinant for functions belonging to such class.

Motivated by the classes $S_s^{*}M(p,\alpha,\beta,\delta)$ and $M_s(\lambda)$, we develop the following subclass of $T(p)$ which is denoted by $S_sT(p,\lambda,\alpha,\beta,\delta)$ with the conditions $p = 1,3,5,7,\ldots$, $0 \le \lambda \le 1$, $0 \le \alpha < 1$, $0 < \beta < 1$, $0 \le \delta < \frac{n-p}{2}$ and $n > p$.

Definition 2.5 A function $f \in S_sT(p,\lambda,\alpha,\beta,\delta)$ if it satisfies

$$\left|\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)[f(z)-f(-z)]+\lambda z[f(z)-f(-z)]'}-(p+\delta)\right|$$
$$<\beta\left|\frac{\alpha[zf'(z)+\lambda z^2 f''(z)]}{(1-\lambda)[f(z)-f(-z)]+\lambda z[f(z)-f(-z)]'}+(p-\delta)\right| \quad (2.1)$$

for $z \in D$.

For different choices of parameters $p,\lambda,\alpha,\beta,$ and δ , we obtain special relationship with the previous known classes as follows:

i) $S_sT(p,0,\alpha,\beta,\delta) \subset S_s^*M(p,\alpha,\beta,\delta)$ which was introduced and studied by (Khairnar and Rajas 2010).

ii) $S_sT(1,0,\alpha,\beta,\delta) \subset M(\alpha,\beta,\delta)$ which was introduced and studied by (Khairnar and More, 2008).

iii) $S_sT(1,0,\alpha,\beta,0) \subset S_s^*T(\alpha,\beta)$ which was introduced and studied by (Halim, Janteng and Darus, 2005) and (Halim, Janteng and Darus, 2007).

iv) $S_sT(1,1,\alpha,\beta,0) \subset C_sT(\alpha,\beta)$ which was introduced and studied by (Wong and Janteng, 2008a) and (Wong and Janteng, 2008b).

In the next section, we will determine the basic properties of functions $f \in S_sT(p,\lambda,\alpha,\beta,\delta)$ such as coefficient estimates, growth and distortion theorem, closure theorem and extreme points.

2.2.1 Preliminary Lemma

The following preliminary lemma is required to prove the main results.

Lemma 2.1 Let $f \in T(p)$, then

$$\sum_{n=1}^{\infty}[\alpha(p+n)[(1-\lambda)+\lambda(p+n)]$$
$$+(p-\delta)[1-(-1)^{p+n}][(1-\lambda)+\lambda(p+n)]]a_{p+n}|z|^n$$
$$<\alpha p[(1-\lambda)+\lambda p]+(p-\delta)[1-(-1)^p][(1-\lambda)+\lambda p]$$

Proof. Let $f \in T(p)$. From (Owa, 1985), we have

$$\textstyle\sum_{n=1}^{\infty}(p+n)\, a_{p+n}|z|^n < p,\ \sum_{n=1}^{\infty}(p+n)^2 a_{p+n}|z|^n < p^2 \text{ and } \sum_{n=1}^{\infty} a_{p+n}|z|^n < 1. \quad (2.2)$$

Thus,

$$\sum_{n=1}^{\infty}\Big[\alpha(p+n)[(1-\lambda)+\lambda(p+n)]$$

$$+(p-\delta)[1-(-1)^{p+n}][(1-\lambda)+\lambda(p+n)]\Big]a_{p+n}|z|^n$$

$$=\sum_{n=1}^{\infty}\alpha(p+n)[(1-\lambda)+\lambda(p+n)]\,a_{p+n}|z|^n$$

$$+\sum_{n=1}^{\infty}(p-\delta)[1-(-1)^{p+n}][(1-\lambda)+\lambda(p+n)]\,a_{p+n}|z|^n$$

$$=\sum_{n=1}^{\infty}\alpha(1-\lambda)(p+n)\,a_{p+n}|z|^n+\sum_{n=1}^{\infty}\alpha\lambda(p+n)^2\,a_{p+n}|z|^n$$

$$+\sum_{n=1}^{\infty}(p-\delta)(1-\lambda)[1-(-1)^{p+n}]\,a_{p+n}|z|^n$$

$$+\sum_{n=1}^{\infty}\lambda(p-\delta)(p+n)[1-(-1)^{p+n}]\,a_{p+n}|z|^n$$

$$=\alpha(1-\lambda)\sum_{n=1}^{\infty}(p+n)\,a_{p+n}|z|^n+\alpha\lambda\sum_{n=1}^{\infty}(p+n)^2\,a_{p+n}|z|^n$$

$$+(p-\delta)(1-\lambda)\sum_{n=1}^{\infty}[1-(-1)^{p+n}]\,a_{p+n}|z|^n$$

$$+\lambda(p-\delta)\sum_{n=1}^{\infty}(p+n)[1-(-1)^{p+n}]\,a_{p+n}|z|^n$$

$$<\alpha(1-\lambda)\sum_{n=1}^{\infty}(p+n)\,a_{p+n}|z|^n+\alpha\lambda\sum_{n=1}^{\infty}(p+n)^2\,a_{p+n}|z|^n$$

$$+(p-\delta)(1-\lambda)\sum_{n=1}^{\infty}[1-(-1)^{p}]\,a_{p+n}|z|^n$$

$$+\lambda(p-\delta)\sum_{n=1}^{\infty}(p+n)[1-(-1)^{p}]\,a_{p+n}|z|^n$$

$$=\alpha(1-\lambda)\sum_{n=1}^{\infty}(p+n)\,a_{p+n}|z|^n+\alpha\lambda\sum_{n=1}^{\infty}(p+n)^2\,a_{p+n}|z|^n$$

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