An Implementation of QSAOR Iterative Method for non-Homogeneous Helmholtz Equations

Jumat Sulaiman
Universiti Malaysia Sabah (UMS)

Mohamed Othman
Universiti Putra Malaysia

Mohana Sundaram Muthuvalu
Universiti Teknologi PETRONAS

International Journal of Contemporary Mathematical Sciences
Vol. 11, 2016, no. 2, 85 - 96
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijcms.2016.51047

An Implementation of QSAOR Iterative Method for Non-Homogeneous Helmholtz Equations

Mohd Kamalrulzaman Md Akhir, Jumat Sulaiman

Faculty of Science and Natural Resources
Universiti Malaysia Sabah
88400 Kota Kinabalu Sabah, Malaysia

Mohamed Othman

Department of Communication Technology and Network
Universiti Putra Malaysia
43400 UPM Serdang Selangor, Malaysia

Mohana Sundaram Muthuvalu

Department of Fundamental and Applied Sciences
Faculty of Science and Information Technology
Universiti Teknologi PETRONAS
Bandar Seri Iskandar, 31750 Tronoh, Perak, Malaysia

Abstract

This paper aims to show the usefulness of the quarter-sweep accelerated over relaxation (QSAOR) method by implementing the quarter-sweep approximation equation based on finite difference (FD) to solve two-dimensional (2D) Helmholtz equations compared to full-sweep accelerated over relaxation (FSAOR) and half sweep accelerated over relaxation (HSAOR) methods. The formulation and implementation of the QSAOR, HSAOR and FSAOR methods are also presented. Some numerical tests were carried out to illustrate that the QSAOR method is superior to HSAOR and FSAOR methods.

Keywords: QSAOR iterative method; Non-homogeneous Helmholtz equations

1 Introduction

With the passage of time, Helmholtz equations are increasingly becoming imperative in numerical analysis for enumerating various problems in different fields. Those fields include engineering and scientific fields, optical waveguide, water wave propagation, acoustic wave scattering, radar scattering and lightwave propagation problems, noise reduction in silencers, time harmonic acoustic and electromagnetic fields [13, 15, 16]. The emphasis of this paper is on Helmholtz equations that are stated by the elliptic equation as shown below:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \alpha U = f(x,y), \tag{1}$$

where Dirichlet boundary conditions and functions $f(x,y)$ are postulated. In this study, it is supposed that the domain is the square unit. It was also supposed that the grid spacing is $h = 1/n$ with $x_i = ih$ and $y_j = jh$, where $(i, j = 1, 2, \ldots, n)$ Eq. (1) can be approximated at point (x_i, y_j) by the full-sweep finite difference (FD) approximation equation that is the most frequently used for approximation. Approximated equation is given below:

$$U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - \left(4 + \alpha h^2\right) U_{i,j} = h^2 f_{i,j}, \tag{2}$$

Using grid spacing $2h$, Eq. (2) could also be discretized using the same formula, resulting in the following formula:

$$U_{i+2,j} + U_{i-2,j} + U_{i,j+2} + U_{i,j-2} - \left(4 + 4h^2\alpha\right) U_{i,j} = 4h^2 f_{i,j}, \tag{3}$$

The clockwise rotation of grid along the $x-y$ axis by 45° results in the following rotated FD approximation equation [1, 4]:

$$U_{i+1,j+1} + U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} - \left(4 + 2h^2\alpha\right) U_{i,j} = 2h^2 f_{i,j} \tag{4}$$

From onwards, three sections are there in this paper. In the first section (Section 2), the formulation of full-, half- and quarter-sweep FD approximation equations based on second order FD schemes is explained. In the second section, the formulations of the QSAOR, FSAOR and HSAOR in solving the system of linear system (LS), attained from discretization of the 2D Helmholtz equations, are elaborated. Lastly, the numerical results and discussions are given in the final section.

2 The AOR Iterative Method

The AOR iterative method was introduced by [2] to investigated the numerical solution of the LS:

$$A\underline{U} = \underline{f}, \tag{5}$$

where $A \in \mathbb{C}^{n,n}$ signifies nonsingular, sparse matrices with non-vanishing diagonal entries, $\underline{U}, \underline{f} \in \mathbb{C}^{n,n}$ and $\underline{U}$ with are to be determined. First, the common AOR method is considered. Let $A \in \mathbb{C}^{n,n}$ be a one-cycle and coherent ordered matrix of the form:

$$A = \begin{bmatrix} D & U \\ U^T & D \end{bmatrix} \tag{6}$$

where $U \in \mathbb{C}^{n1,n2}$, $U^T \in \mathbb{C}^{n2,n1}$ and $D \in \mathbb{C}^{n1,n1}$ are diagonal nonsingular matrices, respectively, with $n_1 + n_2 = n$. So A could be written as follows:

$$A = D - L - V \tag{7}$$

where,

$$D = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ -U^T & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -U \\ 0 & 0 \end{bmatrix}.$$

The Jacobi matrix is defined as:

$$B = D^{-1}(L + V) = \underline{L} + \underline{V} \tag{8}$$

with

$$\underline{L} = D^{-1}L = \begin{bmatrix} 0 & 0 \\ \underline{L} & 0 \end{bmatrix}, \quad \underline{V} = D^{-1}V = \begin{bmatrix} 0 & \underline{V} \\ 0 & 0 \end{bmatrix}.$$

where, $underlineL = -D^{-1}U^T$ and $V = -D^{-1}U$. Generally, the AOR method can be written as:

$$L_{r,w} = rL\big(I - rD^{-1}L\big)^{-1}\left[(1-w)\,I + (w-r)\,L + wD^{-1}V\right] \tag{9}$$

Hypothetically, using Eq. (2) for each point (x_i, y_j) gives a linear system (5) with coefficient matrix A that is given below:

$$A = \begin{bmatrix} D_0 & V_0 & & & \\ L_0 & D_0 & V_0 & & \\ & L_0 & D_0 & \ddots & \\ & & \ddots & \ddots & V_0 \\ & & & L_0 & D_0 \end{bmatrix}_{(n-1)^2 \times (n-1)^2} \tag{10}$$

where

$$D_0 = \begin{bmatrix} R_0 & R_1 & & & \\ R_1^T & R_0 & R_1 & & \\ & R_1^T & R_0 & \ddots & \\ & & \ddots & \ddots & R_1 \\ & & & R_1^T & R_0 \end{bmatrix}_{(n-1)\times(n-1)},$$

$$V_0 = \begin{bmatrix} R_2 & & & & \\ & R_2 & & & \\ & & R_2 & & \\ & & & \ddots & \\ & & & & R_2 \end{bmatrix}_{(n-1)\times(n-1)},$$

$$L_0 = \begin{bmatrix} R_2^T & & & & \\ & R_2^T & & & \\ & & R_2^T & & \\ & & & \ddots & \\ & & & & R_2^T \end{bmatrix}_{(n-1)\times(n-1)}$$

where $\rho_0 = 4 + h^2\alpha$ and the submatrices R_0, R_1 and R_2 are given by

$$R_0 = \begin{bmatrix} \rho_0 & -1 & & -1 \\ -1 & \rho_0 & -1 & 0 \\ & -1 & \rho_0 & -1 \\ -1 & & -1 & \rho_0 \end{bmatrix}, R_1 = \begin{bmatrix} 0 & -1 & & \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ & & & 0 \end{bmatrix}, R_2 = \begin{bmatrix} & & & 0 \\ & & 0 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

Similarly to (10), applying Eq. (4) to each point (x_i, y_j) leads to the LS in (5) with coefficient matrix A given by

$$A = \begin{bmatrix} D_1 & V_1 & & & \\ L_1 & D_1 & V_1 & & \\ & L_1 & D_1 & \ddots & \\ & & \ddots & \ddots & V_1 \\ & & & L_1 & D_1 \end{bmatrix}_{\frac{(n-1)^2}{2}\times\frac{(n-1)^2}{2}} \tag{11}$$

where

$$D_1 = \begin{bmatrix} R_3 & R_4 & & & \\ R_4^T & R_3 & R_4 & & \\ & R_4^T & R_3 & \ddots & \\ & & \ddots & \ddots & R_4 \\ & & & R_4^T & R_3 \end{bmatrix}_{(n-1)\times(n-1)},$$

$$V_1 = \begin{bmatrix} R_5 & & & & \\ & R_5 & & & \\ & & R_5 & & \\ & & & \ddots & \\ & & & & R_5 \end{bmatrix}_{\frac{(n-1)^2}{2}\times\frac{(n-1)^2}{2}},$$

$$L_1 = \begin{bmatrix} R_5^T & & & & \\ & R_5^T & & & \\ & & R_5^T & & \\ & & & \ddots & \\ & & & & R_5^T \end{bmatrix}_{\frac{(n-1)^2}{2}\times\frac{(n-1)^2}{2}}$$

where $\rho_1 = 4 + 2h^2\alpha$ and the submatrices R_3, R_4 and R_5 are given by

$$R_3 = \begin{bmatrix} \rho_1 & -1 & & -1 \\ -1 & \rho_1 & -1 & 0 \\ & -1 & \rho_1 & -1 \\ -1 & & -1 & \rho_1 \end{bmatrix}, R_4 = \begin{bmatrix} 0 & -1 & & \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ & & & 0 \end{bmatrix}, R_5 = \begin{bmatrix} & & & 0 \\ & & 0 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

On the contrary to (10), by applying Eq. (3) to each point (x_i, y_u), we obtain the LS in (5) with coefficient matrix A given by

$$A = \begin{bmatrix} D_2 & V_2 & & & \\ L_2 & D_2 & V_2 & & \\ & L_2 & D_2 & \ddots & \\ & & \ddots & \ddots & V_2 \\ & & & L_2 & D_2 \end{bmatrix}_{(n-2)^2\times(n-2)^2} \tag{12}$$

with

$$D_2 = \begin{bmatrix} R_6 & R_7 & & & \\ R_7^T & R_6 & R_7 & & \\ & R_7^T & R_6 & \ddots & \\ & & \ddots & \ddots & R_7 \\ & & & R_7^T & R_6 \end{bmatrix}_{(n-2)\times(n-2)},$$

$$V_2 = \begin{bmatrix} R_8 & & & & \\ & R_8 & & & \\ & & R_8 & & \\ & & & \ddots & \\ & & & & R_8 \end{bmatrix}_{(n-2)\times(n-2)},$$

$$L_2 = \begin{bmatrix} R_8^T & & & & \\ & R_8^T & & & \\ & & R_8^T & & \\ & & & \ddots & \\ & & & & R_8^T \end{bmatrix}_{(n-2)\times(n-2)}$$

where $\rho_2 = 4 + 4h^2\alpha$ and the submatrices R_6, R_7 and R_8 are given by

$$R_6 = \begin{bmatrix} \rho_2 & -1 & & -1 \\ -1 & \rho_2 & -1 & 0 \\ & -1 & \rho_2 & -1 \\ -1 & & -1 & \rho_2 \end{bmatrix}, R_7 = \begin{bmatrix} 0 & -1 & & \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ & & & 0 \end{bmatrix}, R_8 = \begin{bmatrix} & & & 0 \\ & & 0 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

The coefficient matrices in Eqs. (10), (11) and (12) are point wise tridiagonal with a non-vanishing diagonal element, they have property A and $\pi-$CO, as recommended by [3]. Thus, for full-, half- and quarter-sweep approaches, the theory of point wise AOR iterative method is also valid.

3 The QSAOR Method

Quarter-sweep approach was employed to derive the QSAOR iterative method, in which the domains were divided into three types of points (i.e. $\bullet$, $\square$ and $\circ$), as indicated in Figure 1. By applying the AOR iterative method (9) into Eq. (3), we obtain the QSAOR method for the 2D Helmholtz equation as follows:

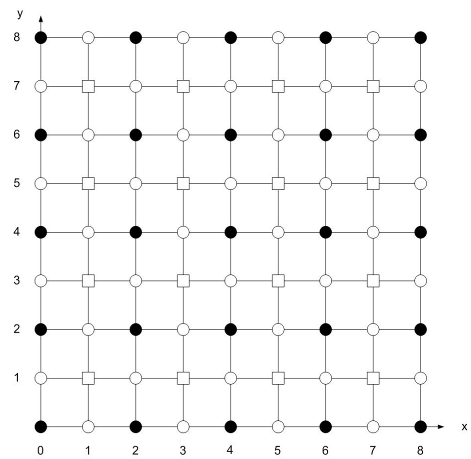

Figure 1: Show the distribution of uniform node points respectively at $n = 8$.

$$U_{i,j}^{(k+1)} = \frac{r}{\rho_2}\left(U_{i-2,j}^{(k+1)} - U_{i-2,j}^{(k)} + U_{i,j-2}^{(k+1)} - U_{i,j-2}^{(k)}\right) + \frac{w}{\rho_2}\left(U_{i+2,j}^{(k)} + U_{i-2,j}^{(k)} + U_{i,j+2}^{(k)} + U_{i,j-2}^{(k)} - 4h^2 f_{i,j}\right) + (1-w)U_{i,j}^{(k)} \quad (13)$$

where $\rho_2 = 4 + 4h^2\alpha$. A quarter of the points, lying on the 2h-grid could be iterated with the help of Eq. (3). In addition, it could be spotted that Eq. (13) includes points of type $\bullet$. Consequently, the iteration could be executed autonomously relating only this type of points. Then, QSAOR iterative method could be formulated as shown in Algorithm 3.1 below.

1. Three types of points (i.e., $\bullet$, $\square$ and $\circ$) should be used to label the solution domain, as depicted in Figure 1.

2. Iterations were performed with the help of Eq. (13) and the values of the values of $r = \omega$ from segment $[1, 2)$ were considered.

3. From the value revealed in step 2, in the interval range of 0.1, the optimal opt having precision of 0.01 was defined by selecting consecutive values, where k was minimal; r was considered to be equal to ω.

4. Using the value of ω opt, experiments were performed and consecutive values of r were chosen with the precision of 0.01 in the interval range of 0.1 from the ω optimal.

5. The value of r optimal was defined, where k was minimal.

6. For the rest of the points, solutions were evaluated by keeping in view the following sequence:

a) Type $\square$ points use the half-sweep FD approximate formula (4) on grid $\sqrt{2h}$, see [6, 7, 8, 9].

$$U_{i,j} = \frac{1}{\rho_2} \left(U_{i+1,j+1} + U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} - 2h^2 f_{i,j}\right)$$

b) Type $\circ$ points use the full-sweep FD approximate formula (2) on grid h, see [10, 11, 12].

$$U_{i,j} = \frac{1}{\rho_0} \left(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - h^2 f_{i,j}\right)$$

7. Finally, the approximate solutions were displayed.

4 Numerical Results

Numerous numerical tests were executed so as to validate the usefulness of the proposed methods. For comparison, three criteria were contemplated for QSAOR, HSAOR and FSAOR methods that includes number of iterations (k), execution time (t) and maximum absolute error (Abs. Error). The tolerance used was $\varepsilon = 10^{-10}$. As mentioned above, FD method was used to discretize and to form the LS for the following problems.

Problem 1

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \alpha U = 4 - \alpha (x-y)^2.$$

$$U(x,0) = x^2, \qquad U(x,1) = (x-2)^2,$$
$$U(0,y) = y^2, \qquad U(1,y) = (y-2)^2.$$

The precise solution is shown below:

$$U(x,y) = (x-y)^2$$

Table 1 tabulates the results of numerical experiments, acquired from executions of the iterative methods in Example 1.

Problem 2

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \alpha U = -(\cos(x+y) + \cos(x-y)) - \alpha \cos(x)\cos(y).$$

$$U(x,0) = \cos y, \qquad U\left(x, \frac{\pi}{2}\right) = 0,$$
$$U(0,y) = \cos x, \qquad U(\pi, y) = -\cos y.$$

The particular solution is given below:

$$U(x, y) = \cos(x)\cos(y).$$

For Example 2 , Table 1 records numerical results of FSAOR, HSAOR and QSAOR iterative methods.

5 Discussion

In the Helmholtz equation model, three types of point wise AOR iterative methods were implemented in order to monitor the number of iterations and the execution times. It could be found from the numerical results that among the other two AOR iterative methods (FSAOR and HSAOR), QSAOR methods are the speediest with regards to the execution time or the number of iterations. In addition, if the computational complexity of all three AOR iterative methods is compared, then the QSAOR method bears the least computational complexity.

Practically, the QSAOR methods are as accurate as the HSAOR and FSAOR iterative methods, however, they benefit in terms of less computing times and reduced number of iterations in order to accomplish the same outcomes. For instance, in Problems 1 and 2, the number of iterations of QSAOR reduced about $46\% - 52\%$ and $43\% - 70\%$, and $22\% - 31\%$ and $22\% - 39\%$, in comparison with FSAOR and HSSOR iterative methods respectively. Furthermore, the execution times of QSAOR were only about $64\% - 73\%$ and $63\% - 84\%$, and $15\% - 21\%$ and $30\% - 32\%$, compared to FSAOR and HSAOR iterative methods in Problems 1 and 2 respectively. Thus, auspicious outcomes are obtained from the experimental results, which establishes the fact that they could be implemented as a substitute for the conventional FD scheme.

From the obtained number of iterations and timings, it could be observed that the QSAOR iterative method requires the least time for all n among the three AOR iterative methods. This is because, among the three iterative methods, the QSAOR iterative method requires the least number of iterations and computational operations.

Besides, the accuracy is considerably improved, because all the methods utilized the descriptive stencil $O(h^2)$. As a whole, the numerical results illustrate that the QSAOR iterative method is superior to the FSAOR and HSAOR iterative methods. This is due to the computational complexity of the QSAOR iterative method, which was reduced around 75% and 50% for FSAOR and HSAOR iterative methods respectively

Table 1: Number of iterations, execution time (seconds) and maximum absolute error for the iterative methods for Examples 1 and 2.

Example 1						
n	Methods	r	ω	k	t	Abs.Error
64	FSAOR	1.89	1.88	206	0.06	1.8589e-10
	HSAOR	1.84	1.83	165	0.04	6.7526e-10
	QSAOR	1.79	1.74	104	0.02	3.4266e-10
128	FSAOR	1.95	1.90	432	0.33	8.8315e-10
	HSAOR	1.93	1.82	331	0.17	2.5143e-10
	QSAOR	1.89	1.88	206	0.07	1.8589e-10
256	FSAOR	1.97	1.97	798	2.14	3.6376e-9
	HSAOR	1.96	1.94	590	1.10	5.5464e-10
	QSAOR	1.94	1.94	433	0.48	1.9934e-9
512	FSAOR	1.99	1.98	2081	25.54	1.0417e-9
	HSAOR	1.97	1.96	1178	12.47	7.6380e-10
	QSAOR	1.97	1.97	798	3.22	3.6376e-9
Example 2						
n	Methods	r	ω	k	t	Abs.Error
64	FSAOR	1.88	1.90	263	0.15	1.4488e-6
	HSAOR	1.85	1.80	151	0.09	5.7996e-6
	QSAOR	1.79	1.76	114	0.04	5.7943e-6
128	FSAOR	1.94	1.94	472	0.45	3.6134e-7
	HSAOR	1.92	1.91	297	0.20	1.4505e-6
	QSAOR	1.89	1.88	218	0.13	1.4498e-6
256	FSAOR	1.97	1.96	874	3.05	8.8714e-8
	HSAOR	1.96	1.95	583	1.53	3.6212e-8
	QSAOR	1.94	1.94	472	0.65	3.6134e-7
512	FSAOR	1.99	1.98	2083	32.16	2.3383e-8
	HSAOR	1.98	1.99	1259	20.69	9.0578e-8
	QSAOR	1.97	1.96	874	4.88	8.8714e-8

References

[1] A.R. Abdullah, The Four Point Explicit Decoupled Group (EDG) Method: A Fast Poisson Solver, *Intern. J. of Comp. Math.*, **38** (1991), 61-70.

[2] A. Hadjidimos, Accelerated overrelaxation method, *Math. of Comput.*, **32** (1978), 149-157. http://dx.doi.org/10.1090/s0025-5718-1978-0483340-6

[3] D.M. Young, *Iterative Solution of Large Linear Systems*, London, Academic Press, 1971.

[4] G. Dahlquist., and A. Bjorck, *Numerical Method*, Prentice Hall, Englewood Cliffs, New Jersey, 1974.

[5] M.K.M. Akhir, M. Othman, J. Sulaiman, Z.A. Majid, and M. Suleiman, Numerical Solutions of Helmholtz Using New Four Point EGMSOR Iterative Method equations, *App. Math. Scien.*, **80** (2011), 3991-4004.

[6] M.K.M. Akhir., M. Othman, J. Sulaiman, Z.A. Majid, and M. Suleiman, Half-Sweep Modified Successive Over-Relaxation for Solving Two-Dimensional Helmholtz Equations, *Aus. J. of. Bas. and App. Scien.*, **12** (2011), 3033-3039.

[7] M.K.M. Akhir., M. Othman, J. Sulaiman, Z.A. Majid, and M. Suleiman, The Four Point-EDGMSOR Iterative Method for Solution of 2D Helmholtz Equations, *Lecture Notes In Communications in Computer and Information Science, A. Abd Manaf, Springer-Verlag Berlin Heidelberg,* Berlin, **253** (2011), 218-227.
http://dx.doi.org/10.1007/978-3-642-25462-8_19

[8] M.K.M. Akhir, M. Othman, J. Sulaiman, Z.A. Majid, and M. Suleiman, Half Sweep Iterative Method for Solving Two-Dimensional Helmholtz Equations, *Int. J. of App. Math. and Stat.*, **29** (2012), 101-109.

[9] M.K.M. Akhir, M. Othman, J. Sulaiman, Z.A. Majid, and M. Suleiman, Four Point Explicit Decoupled Group Iterative Method Applied to Two-Dimensional Helmholtz Equation, *Int. J. of Math. Analy.*, **29** (2012), 963-974.

[10] M.K.M. Akhir, M. Othman, J. Sulaiman, Z.A. Majid, and M. Suleiman, An Efficient Quarter-Sweep Modified Successive Over-Relaxation for Solving Two-Dimensional Helmholtz Equations, *Int. J. of Pu. and App. Math.*, **80** (2012), 751-764.

[11] M.K.M. Akhir, J. Sulaiman, M. Othman, Z.A. Majid, Muthuvalu, M.S and E.Aruchunan, QSMSOR Iterative Method for the Solution of 2D Homogeneous Helmholtz Equation, *Int. J. of Math. Analy.*, **51** (2014), 2493-2505. http://dx.doi.org/10.12988/ijma.2014.48237

[12] M.K.M. Akhir, J. Sulaiman, M. Othman, Z.A. Majid, Muthuvalu, M.S and E.Aruchunan, Application of Four-Point MEGMSOR Method for the Solution of 2D Helmholtz Equations, *Int. J. of Math. Analy.*, **38** (2015), 1847-1862. http://dx.doi.org/10.12988/ijma.2015.56170

[13] M. Nabavi., M.H.K Sidiqui, and J. Daraghi, A new 9-point sixth-order accurate compact finite difference method for Helmholtz equation, *J. of S. and Vib.*, **307** (2007), 972-982. http://dx.doi.org/10.1016/j.jsv.2007.06.070

[14] M. Othman., and A.R. Rahman, An Efficient A New Technique Point Iterative Method for Solving Poisson Equation on MIMD Computer System, *S. Mal.*, (1998), 101-117.

[15] M. Yokota., and N. Sugio, Application of Multigrid Method to Lightwave Propagation in Two-Dimensional Optical Waveguide, *J. of Electr. and Comm.*, **86** (2002), 1168-1175.

[16] N.M. Kassim., A.B. Mohammad, and M.N. Ibrahim, Optical Waveguide Modelling Based on Scalar Finite Difference Scheme, *Jurnal Teknologi*, **45** (2006), 181-194. http://dx.doi.org/10.11113/jt.v45.328

Received: October 28, 2015; Published: December 21, 2015