ON THE APPROXIMATE SOLUTIONS OF SYSTEMS OF ODEs BY LEGENDRE OPERATIONAL MATRIX OF DIFFERENTIATION

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Abstract. In this article, a general framework for solving system of ordinary differential equations by implementing a relatively new numerical technique called the Legendre operational matrix of differentiation is presented for the first time. This method can be an effective procedure to obtain analytic and approximate solutions for different systems of ordinary differential equations. Different from other numerical techniques, shifted Legendre polynomials and their properties are employed for deriving a general procedure for forming this matrix. Comparisons are made between approximate solutions, exact solutions and numerical ones for several examples. Moreover, estimate error for the given algorithm is presented.

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1. Introduction

Many real life phenomena can be modelled by systems of ordinary differential equations (ODEs). For example, mathematical models of series of circuits and mechanical systems involving several springs attached in series can lead to a system of differential equations. Moreover, such systems are often encountered in chemical, ecological, biological, and engineering applications [13]. Many phenomena in chemical kinetics and engineering are modelled by stiff systems [7]. Stiff problems pose special computational difficulties because explicit numerical methods cannot solve these problems without severe limitations on the step size [2]. In control theory, system of ODEs can also exhibit chaotic behaviours [11, 10]. As is well-known, a chaotic system is a nonlinear deterministic system having complex and unpredictable behavior and exhibits sensitive dependence on initial conditions. One such system is the so-called Genesio system [6]. It is one of the paradigms of chaos since it captures many features of chaotic systems.

A system of ODEs can be expressed in the form:

$$y_i' = f_i(t, y_1, \ldots, y_n), \quad y_i(t_0) = y_{0,i}, \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where f_i are (linear or non-linear) real-valued functions, t_0 and $y_{0,i}$ are real numbers. By a change of independent variable $t \to t + t_0$, systems of the form (1.1) can always be translated to the origin, and so in this paper, we focus on finding approximate solution to equations of the form

$$y_i' = f_i(t, y_1, \ldots, y_n), \quad y_i(0) = y_{0,i}, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

Various numerical integration algorithms (for example, Runge-Kutta algorithms) for approximating solutions of the above of systems (1.2) have been presented in the literature. However, these algorithms offer approximate solutions at discrete points only thereby making it impossible to get continuous solutions. Approximate analytic solutions of certain classes of systems of ODEs have been given in [8], [2].

Legendre operational matrix of differentiation (LOMD), first proposed by Saadatmandi and Dehghan [12], is a powerful method for solving linear and non-linear problems. They extended the application of Legendre polynomials to solve fractional differential equations. Pandey et al. [9] employed LOMD to solve Lane-Emden type equations. Recently, the method was applied for several modeled [1], [3]. Different from the Taylor series the Legender series approximation can update the all the previous coefficients when the new term is added. Moreover, the Legender approximation is more rapidly than the Taylor approximation with order of convergent $1/n!$ [5]. That leads to give more accurate solution. To the best of the authors' knowledge, the present work demonstrates for the first time the applicability of LOMD for obtaining the approximate analytic solutions of the system of the form (1.2). A general framework for solving system of ODEs via LOMD is presented. Several examples are studied to demonstrate the accuracy of the method.

2. Legendre polynomials and operational matrix of differentiation

The mth-order Legendre polynomials, $L_m(z)$, on the interval $[-1,1]$ are defined as

$$\begin{aligned} L_0(z) &= 1, \\ L_1(z) &= z, \\ L_{m+1}(z) &= \frac{2m+1}{m+1}\, z\, L_m(z) - \frac{m}{m+1}\, z\, L_{m-1}(z), \quad m = 1,2,\ldots. \end{aligned}$$

These polynomials on the interval $z \in [0,1]$, so-called shifted Legendre polynomials, can be defined by introducing the change of variable $z = 2t-1$. The shifted Legendre polynomials $L_m(2t-1)$ denoted by $P_m(t)$ can be obtained as

$$P_{m+1}(t) = \frac{(2m+1)(2t-1)}{(m+1)}\, P_m(t) - \frac{m}{m+1}\, P_{m-1}(t), \quad m = 1,2,\ldots,$$

where $P_0(t) = 1$ and $P_1(t) = 2t-1$. The analytic form of the shifted Legendre polynomial $P_m(t)$ of degree m is given by

$$P_m(t) = \sum_{i=1}^{m} (-1)^{m+i} \frac{(m+i)\,!\, t^i}{(m-i)\,(i\,!)^2}. \tag{2.1}$$

Note that $P_m(0) = (-1)^m$ and $P_m(1) = 1$ satisfy the orthogonality condition

$$\int_0^1 P_m(t) P_j(t)\,\mathrm{d}t = \begin{cases} \frac{1}{2m+1} & \text{for} \quad m = j, \\ 0 & \text{for} \quad m \neq j. \end{cases}$$

A function $y(t)$ square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as

$$y(t) = \sum_{j=0}^{\infty} c_j P_j(t), \tag{2.2}$$

where the coefficients c_j are given by $c_j = (2j+1)\int_0^1 y(t)P_j(t)\,\mathrm{d}t,\ j = 1,2,\ldots.$

In practice, we consider the $(m+1)$-term shifted Legendre polynomial so that

$$y(t) = \sum_{j=0}^{m} c_j P_j(t) = \mathrm{C}^{\mathrm{T}}\phi(t), \tag{2.3}$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\phi(t)$ are given by

$$\mathrm{C}^{\mathrm{T}} = [\mathrm{c}_0, \mathrm{c}_1, \ldots, \mathrm{c}_m], \quad \phi(t) = [P_0(t), P_1(t), \ldots, P_m(t)]^{\mathrm{T}}.$$

The derivative of the vector $\phi(t)$ can be expressed as

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = \mathrm{D}^1\phi(t),\ \frac{\mathrm{d}^2\phi(t)}{\mathrm{d}t^2} = (\mathrm{D}^1)^2\phi(t),\ \ldots,\ \frac{\mathrm{d}^n\phi(t)}{\mathrm{d}t^n} = (\mathrm{D}^1)^n\phi(t), \tag{2.4}$$

where D^1 is the $(m+1)\times(m+1)$ operational matrix of derivative. A general method of constructing such operational matrix of derivative could be presented as follows:

1. Differentiate analytically some polynomials of first degree.
2. Express these derivatives as a linear combination of polynomials of lower degree.
3. Find a general formula.

Now, the general formula of the operational matrix of derivative D^1 is given by

$$D^1 = (d_{ij}) = \begin{cases} 2(2j+1), & \text{for } j = i-k, \\ \quad 0 & \text{Otherwise.} \end{cases} \quad \begin{cases} k = 1,3,\ldots,m, & \text{if } m \text{ odd}, \\ k = 1,3,\ldots,m-1, & \text{if } m \text{ even}, \end{cases}$$

For example, for odd m we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & 10 & 0 & \ldots & (2m-3) & 0 & 0 \\ 0 & 6 & 0 & 14 & \ldots & 0 & (2m-1) & 0 \end{pmatrix}. \tag{2.5}$$

3. Applications of the operational matrix of derivative

To solve (1.2) by means of the operational matrix of derivative [12, 4], we approximate $(y_i(t))^m$ and $g_i(t)$ by the shifted Legendre Polynomials as

$$(y_i(t))^m \simeq \left(C_i^T\phi(t)\right)^m, \tag{3.1}$$

$$g_i(t) \simeq G_i^T\phi(t), \tag{3.2}$$

where the vector $G^T = [g_0(t),\ldots,g_m(x)]^T$ represents the non-homogenous term. By using equations (2.4), (3.1) and (3.2) we have

$$y_i'(t) \simeq C_i^T(D^1)\phi(t). \tag{3.3}$$

Employing equation (3.1), the residual $\Re(x)$ for equation (1.2) can be written as

$$\Re(t) \simeq C_i^T(D^1)\phi(t) - f(t, C_1^T\phi(t),\ldots,C_m^T\phi(t)) - G_i^T\phi(t). \tag{3.4}$$

Now, to find the solution $y_i(t)$ given in equation (2.3), we first collocate equation (3.4) at m points. For suitable collocation points we use the first m shifted Legendre roots of $P_{m+1}(x)$. These equations together with initial condition generate $(m+1)$ nonlinear equations which can be solved using Newton's iterative method. Consequently, $y_i(t)$ given in equation (2.3) can be calculated.

3.1. Error estimate

In this section, we give an error estimate for the LOM solution. Firstly, we define the exact and the approximate solution $y_i, \bar{y_i}$ respectively, and the error $e_i = y_i - \bar{y_i}$. The residual error of the approximate solution given by

$$R_i = \bar{y_i}' - f_i(t, \bar{y_1}, \bar{y_2}, \ldots \bar{y_n}). \tag{3.5}$$

If $R_i = 0$, then the solution is exact; else, taking (1.2)–(3.5) yields

$$e_i' - f_i(t, y_1, y_2, \ldots y_n) + f_i(t, \bar{y_1}, \bar{y_2}, \ldots \bar{y_n}) + R_i = 0,$$

which is differential equation with initial conditions $y_i(0) = 0$ for $i = 1, 2, \ldots, n$.

4. Numerical experiments

We demonstrate the efficiency of the present method through some illustrative examples.

4.1. Example 1

Let us consider the following linear system of ODEs [8]:

$$y_1'(t) = y_1(t) + y_2(t), \tag{4.1}$$
$$y_2'(t) = -y_1(t) + y_2(t), \tag{4.2}$$

with initial condition

$$y_1(0) = 0, \quad y_2(0) = 1. \tag{4.3}$$

The exact solution is $y_1(t) = e^t \sin(x), \quad y_2(t) = e^t \cos(x)$.

First, we consider the solution of the form

$$y_1(t) = \mathrm{C}_1^{\mathrm{T}} \phi(t), \tag{4.4}$$
$$y_2(t) = \mathrm{C}_2^{\mathrm{T}} \phi(t). \tag{4.5}$$

Thus, the solution will be in the form

$$y_1(t) = c_0\, P_0(t) + c_1\, P_1(t) + \ldots + c_m\, P_m(t) = \mathrm{C}_1^{\mathrm{T}} \phi(t), \tag{4.6}$$
$$y_2(t) = h_0\, P_0(t) + h_1\, P_1(t) + \ldots + h_m\, P_m(t) = \mathrm{C}_2^{\mathrm{T}} \phi(t). \tag{4.7}$$

By applying the initial conditions, we have

$$c_0 - c_1 + \ldots + c_m = 1$$
$$h_0 - h_1 + \ldots + c_m = 0.$$

The residual error of the solution can be written as

$$\Re_1 = C_1^{\mathrm{T}} D^1 \phi(t) - C_1^{\mathrm{T}} \phi(t) - C_2^{\mathrm{T}} \phi(t), \tag{4.8}$$
$$\Re_2 = C_2^{\mathrm{T}} D^1 \phi(t) + C_1^{\mathrm{T}} \phi(t) - C_2^{\mathrm{T}} \phi(t). \tag{4.9}$$

Now, taking the first m roots of $m+1$ shifted Legendre polynomials and substituted in (4.8) and (4.9) which equal to zero. By this way, we can have $2m+2$ algebraic equations with $2m+2$ constants which can be easily solved by Newton's iterative method. By taking $m=6$, the values of C_1 and C_2 are

$$(4.10)\quad C_1 = \begin{pmatrix} 0.9093308 \\ 1.134074 \\ 0.2363245 \\ 0.9902234\times 10^{-2} \\ -0.1957476\times 10^{-2} \\ -0.2967196\times 10^{-3} \\ -0.1815798\times 10^{-4} \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1.378024 \\ 0.2720071 \\ -0.1402875 \\ -0.03758258 \\ -0.3404238\times 10^{-2} \\ -0.8353643\times 10^{-4} \\ 0.8351776\times 10^{-5} \end{pmatrix}.$$

The computed and exact solutions for the case $m=6$ are plotted on the same graph in Figure 1.

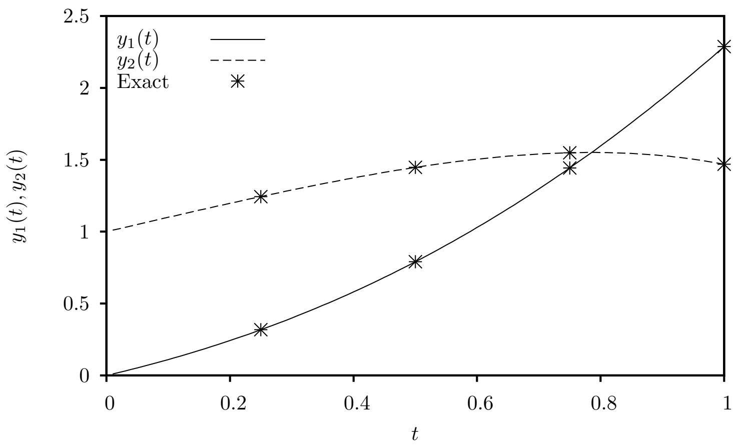


Figure 1: The LOMD solution with the exact one for Example 1 with $m=6$.

Moreover, the absolute errors between the exact solution and the present ones for $m=6$ and $m=8$ are given in Figure 2.

Clearly, the figure shows that increasing m decreases the absolute errors. The estimate error can be given as

$$(4.11)\quad e_1' - e_1 - e_2 + R_1 = 0,$$

$$(4.12)\quad e_2' + e_1 - e_2 + R_2 = 0.$$

Subject to the initial conditions $e_1(0)=0, e_2(0)=0$.

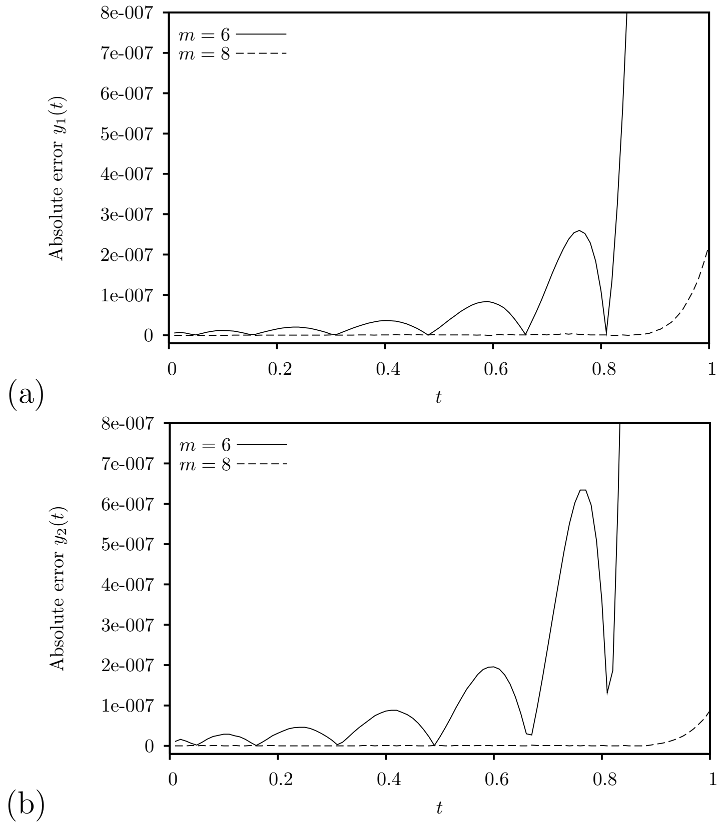


Figure 2: The absolute errors between the LOMD solutions and exact one for Example 1.

4.2. Example 2

Consider the nonlinear stiff system of ODE [2]

$$y_1'(t) = -1002y_1(t) + 1000y_2^2(t), \tag{4.13}$$
$$y_2'(t) = y_1(t) - y_2(t) - y_2^2(t), \tag{4.14}$$

subject to the initial conditions

$$y_1(0) = 1, \quad y_2(0) = 1. \tag{4.15}$$

The exact solution is

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{-t}.$$

By assuming the solution of the form

$$y_1(t) = c_0\, P_0(t) + c_1\, P_1(t) + \ldots + c_m\, P_m(t) = \mathrm{C}_1^{\mathrm{T}}\phi(t), \tag{4.16}$$
$$y_2(t) = h_0\, P_0(t) + h_1\, P_1(t) + \ldots + h_m\, P_m(t) = \mathrm{C}_2^{\mathrm{T}}\phi(t), \tag{4.17}$$

and applying the initial conditions, we have

$$\begin{aligned} c_0 - c_1 + \ldots + c_m &= 1 \\ h_0 - h_1 + \ldots + c_m &= 0. \end{aligned}$$

The residual errors are

$$\Re_1 = C_1^{\mathrm{T}} D^1 \phi(t) + 1002 C_1^{\mathrm{T}} \phi(t) - 1000 (C_2^{\mathrm{T}} \phi(t))^2, \quad (4.18)$$
$$\Re_2 = C_2^{\mathrm{T}} D^1 \phi(t) - C_1^{\mathrm{T}} \phi(t) + C_2^{\mathrm{T}} \phi(t) + (C_2^{\mathrm{T}} \phi(t))^2. \quad (4.19)$$

Using the same manner as in the previous example, we can create $2m+2$ algebraic equation with $2m+2$ constant, which can be easily solved via Newton's iterative method. By taking $m = 8$, the value of $\mathrm{C}_1^{\mathrm{T}}$ and $\mathrm{C}_2^{\mathrm{T}}$ are

$$\mathrm{C}_1 = \begin{pmatrix} 0.4323324 \\ -0.4060059 \\ 0.1316326 \\ -0.02591911 \\ 0.3665828 \times 10^{-2} \\ -0.4044962 \times 10^{-3} \\ 0.3663535 \times 10^{-4} \\ -0.2858631 \times 10^{-5} \\ 0.2868407 \times 10^{-6} \end{pmatrix}, \quad \mathrm{C}_2 = \begin{pmatrix} 0.6321206 \\ -0.3109150 \\ 0.05145307 \\ -0.5125022 \times 10^{-2} \\ 0.3650665 \times 10^{-3} \\ -0.2024552 \times 10^{-4} \\ 0.1016656 \times 10^{-5} \\ -0.3359670 \times 10^{-7} \\ 0.1173426 \times 10^{-8} \end{pmatrix}. \quad (4.20)$$

The given solution with exact one are presented in Figure 3. Moreover, the absolute errors between the exact solution and the present one for $m = 6$ and $m = 8$ are given in Figure 4. The estimate error equations given as

$$e_1' + 1002 e_1 - 1000(e_2^2 + 2 e_2 \bar{y}_2) + R_1 = 0 \quad (4.21)$$
$$e_2' - e_1 + e_2 + e_2^2 + 2 e_2 \bar{y}_2 + R_2 = 0 \quad (4.22)$$

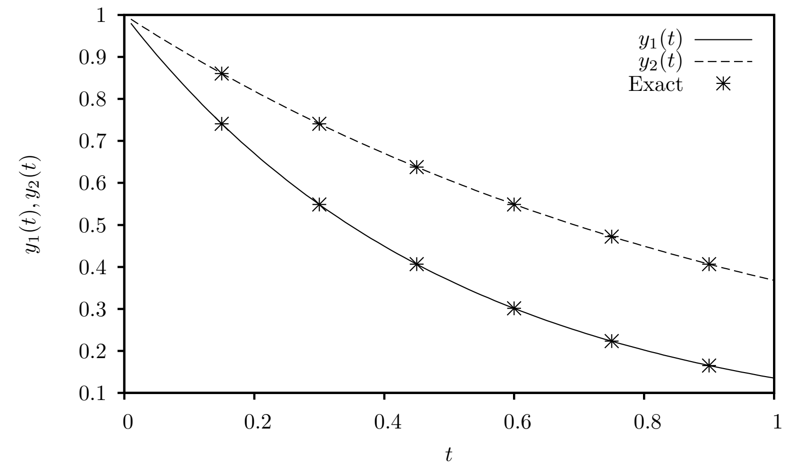
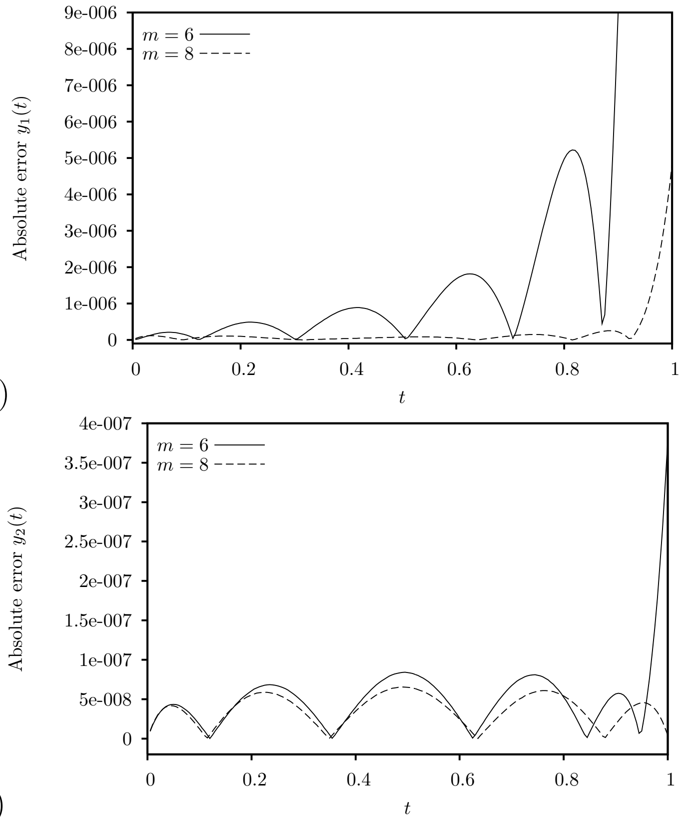


Figure 3: The LOMD solution with exact one for Example 2 with $m = 8$.



(a)

(b)

Figure 4: The absolute error between the LOMD solution and exact one for Example 2.

4.3. Example 3

Finally, we consider the nonlinear Genesio system [2]

$$y_1'(t) = y_2(t), \tag{4.23}$$

$$y_2'(t) = y_3(t), \tag{4.24}$$

$$y_3'(t) = -cy_1(t) - by_2(t) - ay_3(t) + y_1(t)^2, \tag{4.25}$$

subject to the initial conditions

$$y_1(0) = 0.2 \qquad y_2(0) = -0.3, \qquad y_3(0) = 0.1, \tag{4.26}$$

where a, b and c are positive constants, satisfying $ab < c$. The Genesio system includes a simple square part and three simple ordinary differential equations that depend on three positive real parameters [2].

According to the framework of the LOMD, the solution can be expanded as series of shifted Legendre polynomials

$$y_1(t) = c_0\, P_0(t) + c_1\, P_1(t) + \ldots + c_m\, P_m(t) = \mathrm{C}_1^{\mathrm{T}}\phi(t), \tag{4.27}$$

$$y_2(t) = h_0\, P_0(t) + h_1\, P_1(t) + \ldots + h_m\, P_m(t) = \mathrm{C}_2^{\mathrm{T}}\phi(t), \tag{4.28}$$

$$y_3(t) = s_0\, P_0(t) + s_1\, P_1(t) + \ldots + s_m\, P_m(t) = \mathrm{C}_3^{\mathrm{T}}\phi(t). \tag{4.29}$$

Substituting (4.26) into (4.27–4.29) we have

$$\begin{aligned} c_0 - c_1 + \ldots + c_m &= 0.2, \\ h_0 - h_1 + \ldots + h_m &= -0.3, \\ s_0 - s_1 + \ldots + s_m &= 0.1. \end{aligned}$$

The residual errors are then given by

$$\Re_1 = \mathrm{C}_1^{\mathrm{T}} D^1 \phi(t) - \mathrm{C}_2^{\mathrm{T}} \phi(t), \tag{4.30}$$

$$\Re_2 = \mathrm{C}_2^{\mathrm{T}} D^1 \phi(t) - \mathrm{C}_3^{\mathrm{T}} \phi(t), \tag{4.31}$$

$$\Re_3 = \mathrm{C}_3^{\mathrm{T}} D^1 \phi(t) + c\mathrm{C}_1^{\mathrm{T}} \phi(t) + b\mathrm{C}_2^{\mathrm{T}} \phi(t) + a\mathrm{C}_3^{\mathrm{T}} \phi(t) - (\mathrm{C}_1^{\mathrm{T}} \phi(t))^2. \tag{4.32}$$

Now, taking the first m roots of $m+1$ shifted Legendre polynomials and substituting in (4.30) and (4.32) which equals to zero, we have $3m+3$ algebraic equations with $3m+3$ constants which can be easily solved by Newton's iterative method.

The values of C_1, C_2 and C_3 with $m = 8$ are

$$\mathrm{C}_1 = \begin{pmatrix} 0.06305404 \\ -0.1292413 \\ 0.9418267 \times 10^{-2} \\ 0.2318345 \times 10^{-2} \\ 0.5280416 \times 10^{-3} \\ -0.7923437 \times 10^{-4} \\ -0.2417998 \times 10^{-5} \\ 0.1845305 \times 10^{-6} \\ 0.2605319 \times 10^{-7} \end{pmatrix}, \quad \mathrm{C}_2 = \begin{pmatrix} -0.2540041 \\ 0.05966350 \\ 0.02239294 \\ 0.7359095 \times 10^{-2} \\ -0.1422908 \times 10^{-2} \\ -0.5263004 \times 10^{-4} \\ 0.4794184 \times 10^{-5} \\ 0.7757259 \times 10^{-6} \\ -0.2767657 \times 10^{-8} \end{pmatrix}, \tag{4.33}$$

$$\mathrm{C}_3 = \begin{pmatrix} 0.1339414 \\ 0.1258488 \\ 0.07307226 \\ -0.01985379 \\ -0.9335802 \times 10^{-3} \\ 0.1052177 \times 10^{-3} \\ 0.2000128 \times 10^{-4} \\ -0.2154617 \times 10^{-6} \\ -0.7247838 \times 10^{-7} \end{pmatrix}. \tag{4.34}$$

Both the LOMD and the numerical solutions obtained by the 4th-order Runge-Kutta (RK4) method for Genesio system are plotted in Figure 5.

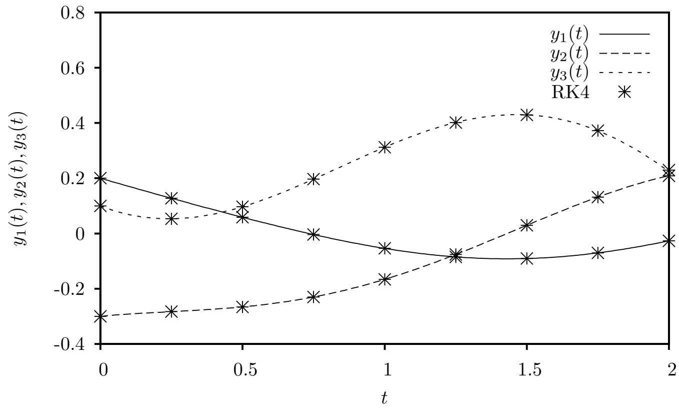


Figure 5: The solution of Genesio system with RK4 solution when $m = 10$.

It is clear that the LOMD solutions agree well with that of the RK4 solutions (see also Figure 6). The estimate error equations given as

$$e_1' - e_1 + R_1 = 0, \tag{4.35}$$

$$e_2' - e_3 + R_2 = 0, \tag{4.36}$$

$$e_3' - ce_1 + be_2 + ae_3 - e_1^2 - 2e_1\bar{y}_1 + R3 = 0. \tag{4.37}$$

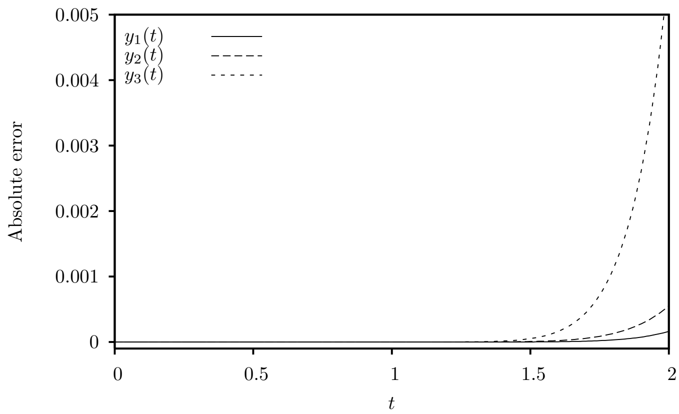


Figure 6: The absolute error between LOMD with $m = 10$ and the RK4 solution.

It is worth mentioning that the estimate error and the absolute error are closed for this example.

5. Conclusions

The Legender polynomial operational matrix of differentiation has been applied to solve linear and nonlinear systems of ordinary differential equations. The advantage of the method over others is that only small size operational matrix is required to provide the solution of high accuracy. The obtained solutions for various examples demonstrate the validity and applicability of the method compared to the other existing methods.

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