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HSAOR Iterative Method for the Finite Element Solution of 2D Poisson Equations

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ABSTRACT

This paper deliberates the use of the Half-sweep Accelerated Overelaxation (HSAOR) method to solve 2D Poisson equations by using the half-sweep triangle finite element (FE) approximation equation based on the Galerkin scheme. In fact, formulations of the full sweep successive over relaxation (FSSOR), half sweep successive over relaxation (HSSOR), full-sweep accelerated over relaxation (FSAOR) and half-sweep accelerated over relaxation (HSAOR) triangle finite element (FE) approaches are also shown. Some numerical experiments are steered to show that the HSAOR method is loftier to the existing FSAOR, HSSOR and FSSOR methods.

Keywords: Poisson, AOR method, Galerkin scheme (GS), Triangle Element (TE), half-sweep approach.

2000 Mathematics Subject Classification: 62J12, 62G99.

Computing Classification System: I.4.

1 Introduction

The problem of deciphering partial differential equations ascends in many mathematical models of scientific and engineering applications. One of the most widely held and significant research branches is FE method. Researchers has a broad application in several weighted residual schemes such as the subdomain, collocation, least-square, moments and Galerkin that can be used to setback approximate solutions (Fletcher, 1978; Fletcher, 1984; Belytschko, Krongauz, Organ, Fleming and Krysl, 1996; Zhu, 1999; Yagawa and Furukawa, 2000). Using the first order triangle FE approximation equation based on the Galerkin scheme, this paper proposes the HSAOR for solving the 2D Poisson equation. In comparison, the Full-Sweep Gauss–Seidel (FSGS) iterative methods act as control method. To investigate the effectiveness of the HSAOR iterative method, let us consider the 2D Poisson equation defined as

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x,y), \quad (x,y) \in [a,b] \times [a,b] \tag{1.1}$$

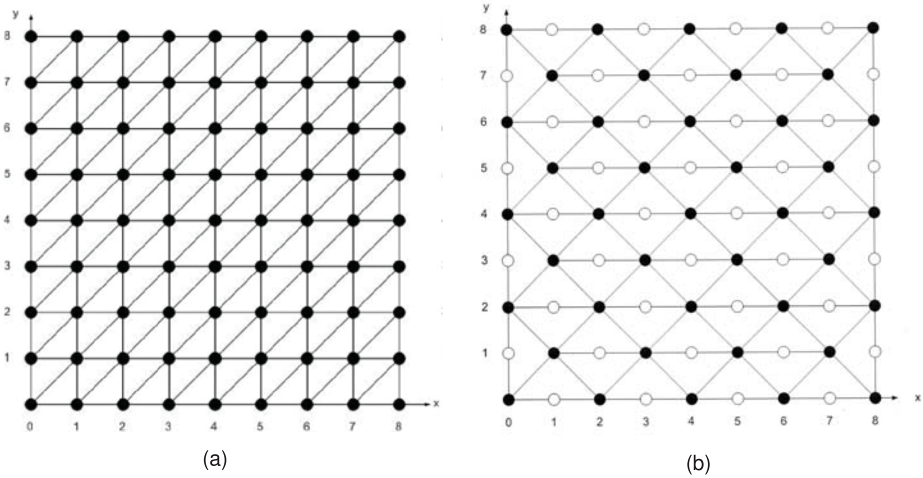


Figure 1: (a) and (b) show the solution domain Ω of triangle elements for the full- and half-sweep cases at $n = 8$.

with the dirichlet boundary conditions

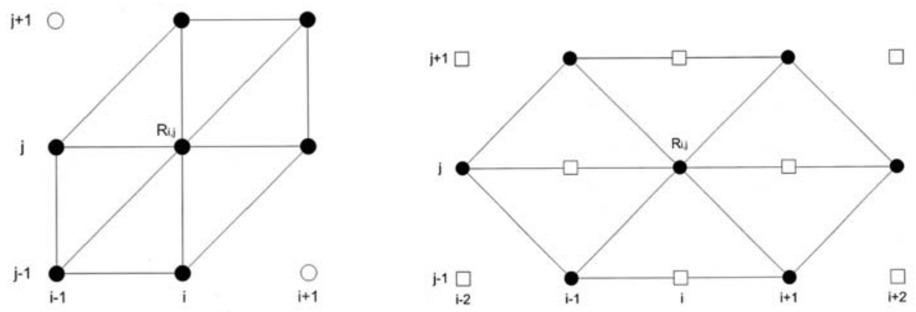
$$\begin{aligned} U(x,a) &= g_1(x), \quad a \le x \le b, \qquad U(x,b) = g_2(x), \quad a \le x \le b, \\ U(a,y) &= g_3(y), \quad a \le x \le b, \qquad U(b,y) = g_4(x), \quad a \le y \le b, \end{aligned}$$

whereas, $f(x,y)$ is a given function with sufficient smoothness. To facilitate in formulating the half-sweep triangle element approximation equations for problem (1.1), our next confab will focus on uniform node points only as shown in Figure. 1. Based on the Figure. 1, the solution domain, need to be discretized uniformly in both x and y directions with a mesh size, h which is defined as

$$h = \frac{b-a}{n}, \quad m = n+1, \tag{1.2}$$

where n is arbitrary positive integer. Based on Figure. 1, we also need to build the networks of triangle FE as a guideline in order to derive triangle FE approximation equations for problem (1.1). Similarly using the same concept of the full-sweep approach applied to FD methods (Abdullah, 1991; Othman and Abdullah, 1998; Sulaiman, Hasan and Othman, 2004), FE networks will consist of several TE in which each will involve three solid node points only of type $\bullet$ as shown in Figure. 1. As a result, the implementation of the full- and half-sweep iterative algorithms will be performed onto the node points of the same type until the iterative convergence criterion will be satisfied. Then other approximate solutions at remaining node points (points of the different type) will be calculated directly (Abdullah, 1991; Evans, 1997; Sulaiman et al., 2004; Yousif and Martins, 2008; Aruchunan and Sulaiman, 2011; Muthuvalu and Sulaiman, 2011; Akhir, Othman, Sulaiman, Majid and Suleiman, 2012).

The outline of this paper is organized in the following way. An implementation of the triangle FE in discretizing problem (1.1) is presented in Section 2 followed by the formulation of the



(a) (b)

Figure 2: (a) and (b) show the definition of the hat function $R_{i,j}(x, y)$, of full- and half-sweep triangle elements at the solution domain.

tested iterative methods in Section 3. Numerical results of the tested iterative methods and concluding remarks are summarized in Section 4 and 5 respectively.

2 Half-Sweep Triangle Element Approximations

Without a loss of platitude, and for vulgarization purpose we will consider case of the FE approximation equation based on the GS to solve 2D Poisson equations. By considering three node points of type • only, the common approximation of the function $U(x, y)$, in the form of interpolation function for an arbitrary triangle element, e is given by (Fletcher, 1978; Fletcher, 1984; Lewis and Ward, 1991).

$$\widetilde{U}^{[e]}(x,y) = N_1(x,y)U_1 + N_2(x,y)U_2 + N_3(x,y)U_3 \tag{2.1}$$

and the shape functions $N_k(x, y)$, $k = 1, 2, 3$, can generally be shown as

$$\det A = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2), \tag{2.2}$$

where,

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} y_2 - y_3 \\ y_3 - y_1 \\ y_1 - y_2 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_3 - x_2 \\ x_1 - x_3 \\ x_2 - x_3 \end{bmatrix}.$$

Adjacent to this, the first order partial derivatives of the shape functions towards x and y are given respectively as

$$\left.\begin{aligned} \frac{\partial}{\partial x}\big(N_k(x,y)\big) &= \frac{b_k}{\det A}, \\ \frac{\partial}{\partial y}\big(N_k(x,y)\big) &= \frac{c_k}{\det A}, \end{aligned}\right\}, \quad k = 1, 2, 3. \tag{2.3}$$

In Figure. 2 (a) and (b), the definition of the hat function, $R_{r,s}(x, y)$ in the solution domain is easily to be shown. Then based on the distribution of the hat function, $R_{r,s}(x, y)$ in the figure, the approximation of the functions, $U(x, y)$ and $f(x, y)$ in case of the full-sweep and half-sweep cases for the intact domain will be well-defined respectively as (Vichnevetsky, 1981)

$$\widetilde{U}(x,y) = \sum_{r=0}^{m}\sum_{s=0}^{m} R_{r,s}(x,y)U_{r,s} \tag{2.4}$$

$$\tilde{f}(x,y) = \sum_{r=0}^{m}\sum_{s=0}^{m} R_{r,s}(x,y)f_{r,s} \tag{2.5}$$

and

$$\widetilde{U}(x,y) = \sum_{r=0,2,4}^{m}\sum_{s=0,2,4}^{m} R_{r,s}(x,y)U_{r,s} + \sum_{r=1,3,5}^{m-1}\sum_{s=1,3,5}^{m-1} R_{r,s}(x,y)U_{r,s} \tag{2.6}$$

$$\tilde{f}(x,y) = \sum_{r=0,2,4}^{m}\sum_{s=0,2,4}^{m} R_{r,s}(x,y)f_{r,s} + \sum_{r=1,3,5}^{m-1}\sum_{s=1,3,5}^{m-1} R_{r,s}(x,y)f_{r,s} \tag{2.7}$$

Thus, Eqs. (2.4) and (2.6) are approximate solutions for problem (1.1). To construct the full-sweep and half-sweep linear FE approximation equations for problem (1.1), this paper proposes the Galerkin scheme. Therefore, let consider the GS (Fletcher, 1978; Fletcher, 1984; Lewis and Ward, 1991) be defined as

$$\iint_D R_{i,j}(x,y)E_{i,j}(x,y) = 0, \quad i,j = 0,1,2,\ldots,m \tag{2.8}$$

where, $E(x, y) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - f(x, y)$ is a residual function. By applying the Green theorem, Eq. (2.5) can be shown in the subsequent form

$$\begin{aligned}\int_\lambda \left(-R_{i,j}(x,y)\frac{\partial u}{\partial y}dx + R_{i,j}(x,y)\frac{\partial u}{\partial x}dy\right)\\ - \int_a^b\int_a^b \left(\frac{\partial R_{i,j}(x,y)}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial R_{i,j}(x,y)}{\partial y}\frac{\partial u}{\partial y}\right)dxdy = F_{i,j}\end{aligned} \tag{2.9}$$

where

$$-\sum\sum K^*_{i,j,r,s}U_{r,s} = \sum\sum C^*_{i,j,r,s}f_{r,s} \tag{2.10}$$

where

$$K^*_{i,j,r,s} = \int_a^b\int_a^b \left(\frac{\partial R_{i,j}}{\partial x}\frac{\partial R_{r,s}}{\partial x}\right)dxdy + \int_a^b\int_a^b \left(\frac{\partial R_{i,j}}{\partial y}\frac{\partial R_{r,s}}{\partial y}\right)dxdy,$$

$$C^*_{i,j,r,s} = \int_a^b\int_a^b \big(R_{i,j}(x,y)R_{r,s}(x,y)\big)dxdy.$$

Actually the linear system in Eq. (2.6) for the full- and half-sweep cases can be easily expressed in the stencil form respectively as follows

Full-sweep:
$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} U_{i,j} = \frac{h^2}{12}\begin{bmatrix} & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & \end{bmatrix} f_{i,j} \tag{2.11}$$

$$\begin{bmatrix} 1 & & 1 & \\ & -4 & & 0 \\ 1 & & 1 & \end{bmatrix} U_{i,j} = \frac{h^2}{6}\begin{bmatrix} 1 & & 1 & \\ & 5 & & 1 \\ 1 & & 1 & \end{bmatrix} f_{i,j}, \quad i = 1$$

$$\text{Half-sweep:} \quad \begin{bmatrix} & 1 & & 1 & \\ 0 & & -4 & & 0 \\ & 1 & & 1 & \end{bmatrix} U_{i,j} = \frac{h^2}{6} \begin{bmatrix} & 1 & & 1 & \\ 1 & & 6 & & 1 \\ & 1 & & 1 & \end{bmatrix} f_{i,j}, \quad i \neq 1, n$$

$$\begin{bmatrix} & 1 & & 1 \\ 0 & & -4 & \\ & 1 & & 1 \end{bmatrix} U_{i,j} = \frac{h^2}{6} \begin{bmatrix} & 1 & & 1 \\ 1 & & 5 & \\ & 1 & & 1 \end{bmatrix} f_{i,j}, \quad i = n \tag{2.12}$$

In fact, the stencil forms in Eqs. (2.11) till (2.12) forms consist of seven node points in formulating their approximation equations. On the other hand, two of its coefficients are zero. Apart of this, the form of the stencil forms for both triangle finite element schemes is the same compared to the existing five points FD scheme, see (Young, 1971; Abdullah, 1991; Yousif and Evans, 1995; Ibrahim and Abdullah, 1995; Evans, 1997; Akhir et al., 2012).

3 The AOR Method

The subsequent discussion can be found in (Hadjidimos, 1978; Evans and Martins, 1994).

3.1 FSAOR Method for Poisson Equation

The subsequent discussion can be found in (Yousif and Martins, 2008).

3.2 HSAOR Method for Poisson Equation

To derive the HSAOR iterative method, we use half-sweep approach, in which the domains are divided into two type of points (i.e., $\bullet$ and $\circ$) as shown in Fig. 1(b). By applying AOR method (Hadjidimos, 1978) into Eq. (2.12), we get the HSAOR method for 2D Poisson equation as

$$\begin{aligned} U_{i,j}^{(k+1)} = \frac{r}{4} \left(U_{i-1,j-1}^{(k+1)} - U_{i-1,j-1}^{(k)} + U_{i+1,j-1}^{(k+1)} - U_{i+1,j-1}^{(k)} \right) \\ + \frac{w}{4} \left(U_{i+1,j+1}^{(k)} + U_{i-1,j-1}^{(k+1)} + U_{i+1,j-1}^{(k+1)} + U_{i-1,j+1}^{(k)} - h^2 F_{i,j} \right) + (1-w) U_{i,j}^{(k)} \end{aligned} \tag{3.1}$$

where

$$F_{i,j} = \frac{1}{12} (f_{i-2,j} + f_{i+2,j} + f_{i-1,j-1} + f_{i-1,j+1} + f_{i+1,j-1} + f_{i+1,j+1} + 6 f_{i,j})$$

Eq. (3.1) allows us to iterate through half of the points, lying on the $\sqrt{2h}$-grid. Again, it can be observed that Eq. (3.1) involves points of type $\bullet$ and $\circ$. Therefore the iteration can be carried out autonomously involving only this type of point. The algorithm of HSAOR method is display in Algorithm 3.2:

Algorithm 3.2. Discretize the solution domain into point of two types $\bullet$ and $\circ$ as shown in Figure 1(b).

1. Perform iterations (using Eq. (3.1)) taking the values of $r = w$ from the segment $[1, 2)$.

2. Within the interval ± 0.1 from the value found in the step 2, define the optimal $w\ opt$ with precision 0.01 by choosing consecutive values for which k is minimal; r is taken the same as w.

3. Perform experiments using the value of $w\ opt$ and choosing consecutive values of r with precision 0.01 within the interval ± 0.1 from the $w\ opt$.

4. Define the value $r\ opt$ for which k is minimal.

5. Evaluate the solutions at the remaining points (Abdullah, 1991; Evans, 1997; Sulaiman et al., 2004; Yousif and Martins, 2008; Aruchunan and Sulaiman, 2011; Muthuvalu and Sulaiman, 2011; Akhir et al., 2012) type ◦ (using Eq. (2.11)).

$$U_{i,j} = \frac{1}{4}(U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - h^2 F_{i,j})$$

where

$$F_{i,j} = \frac{1}{12}(f_{i-1,j} + f_{i+1,j} + f_{i,j-1} + f_{i,j+1} + f_{i-1,j-1} + f_{i+1,j+1} + 6f_{i,j})$$

6. Display approximate solutions.

4 Numerical Results

In this section, Algorithm 3.2 was tested on the following model following 2D Poisson equation:

$$\frac{\partial U}{\partial x^2} + \frac{\partial U}{\partial y^2} = -\cos(x+y) + \cos(x-y) \tag{4.1}$$

where its boundary conditions are given as

$$\begin{aligned} U(x,0) &= \cos x, & U\left(x, \tfrac{\pi}{2}\right) &= 0, \\ U(0,y) &= \cos y, & U(\pi, y) &= -\cos y. \end{aligned}$$

Then exact solution of problem (4.1) is given by

$$U(x,y) = \cos(x)\cos(y).$$

Through the experiments, three parameters were observed, such as the number of iterations (k), maximum absolute error (Abs. Error) and execution time, t (in seconds). Three iterative methods such as Gauss–Seidel (GS), SOR and AOR were tested on several mesh sizes i.e 284, 308, 332 and 356. In the course of implementation the proposed iterative methods, the value of the tolerance error, considered $\varepsilon = 10^{-10}$. The computer language used for the programming is C++, and the program performed on a personal PC Intel(R) Core (TM) i7 CPU 860@3.00 Ghz, 6.00 GB RAM. The operation system used was Windows 7 with the installation Borland C++ compiler version 5.5. Based on the given example, all of the results of numerical experiments are recorded through the implementation of three proposed iterative methods in Table 1. Whereas Table 2 describes the depreciation percentage of the number of iterations and execution time for AOR method compared to the SOR and GS methods.

Table 1: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods.

n	Methods		Methods			Methods			
					k				
				w			r	w	
	FSGS	115954	FSSOR	1.952	3428	FSAOR	1.989	1.979	1890
284	HSGS	60810	HSSOR	1.949	1820	HSAOR	1.961	1.960	1292
	FSGS	134823	FSSOR	1.949	4288	FSAOR	1.989	1.987	1939
308	HSGS	70743	HSSOR	1.941	2528	HSAOR	1.982	1.969	1245
	FSGS	154979	FSSOR	1.939	5960	FSAOR	1.986	1.991	2668
332	HSGS	81316	HSSOR	1.940	2988	HSAOR	1.981	$1..986$	1754
	FSGS	176045	FSSOR	1.939	6815	FSAOR	1.987	1.993	3032
356	HSGS	92654	HSSOR	1.939	3488	HSAOR	1.980	1.989	2215
					t				
	FSGS	267.61	FSSOR	1.952	11.97	FSAOR	1.989	1.979	9.43
284	HSGS	75.85	HSSOR	1.949	5.56	HSAOR	1.961	1.960	4.82
	FSGS	366.50	FSSOR	1.949	17.69	FSAOR	1.989	1.987	11.75
308	HSGS	110.07	HSSOR	1.941	9.52	HSSOR	1.982	1.969	5.94
	FSGS	490.68	FSSOR	1.939	28.58	FSAOR	1.986	1.991	18.84
332	HSGS	153.91	HSSOR	1.940	12.17	HSAOR	1.981	1.986	9.29
	FSGS	638.22	FSSOR	1.939	37.63	FSAOR	1.987	1.993	24.70
356	HSGS	203.54	HSSOR	1.939	17.67	HSAOR	1.980	1.989	5.92
					Abs. Error				
	FSGS	$1.0913e{-6}$	FSSOR	1.952	$3.0217e{-7}$	FSAOR	1.989	1.979	$2.8268e{-7}$
284	HSGS	$4.0311e{-6}$	HSSOR	1.949	$4.0294e{-6}$	HSAOR	1.961	1.960	$4.0294e{-6}$
	FSGS	$1.1936e{-6}$	FSSOR	1.949	$2.6481e{-7}$	FSAOR	1.989	1.987	$2.4128e{-7}$
308	HSGS	$3.4323e{-6}$	HSSOR	1.941	$3.4304e{-6}$	HSAOR	1.982	1.969	$3.4304e{-6}$
	FSGS	$1.3165e{-6}$	FSSOR	1.939	$2.4124e{-7}$	FSAOR	1.986	1.991	$2.0723e{-7}$
332	HSGS	$2.9578e{-6}$	HSSOR	1.940	$2.9558e{-6}$	HSAOR	1.981	$1..986$	$2.9558e{-6}$
	FSGS	$1.4576e{-6}$	FSSOR	1.939	$2.1950e{-7}$	FSAOR	1.987	1.993	$1.8023e{-7}$
356	HSGS	$2.5755e{-6}$	HSSOR	1.939	$2.5734e{-6}$	HSAOR	1.980	1.989	$4.0293e{-6}$

k is the number of iterations, t is the computation time.

Table 2: Reduction percentages of the number of iterations and execution time for the iterative methods compared with FSGS method.

Methods	k	t
HSGS	47.37–473.56	68.11–71.97
FSSOR	96.13–97.04	94.10–95.49
HSSOR	98.01–98.43	97.23–97.86
FSAOR	98.27–98.37	96.13–96.48
HSAOR	98.74–98.89	98.20–99.07

k is the number of iterations, t is the computation time.

5 Discussion of Results

In the previous section, it can be seen that the half- and full-sweep triangle FE approximation equations based on the Galerkin scheme can be easily represented in the stencil forms, see in Eq. (2.11) till (2.12). Through numerical results observed in Table 1, clearly show that by applying the AOR methods can reduce number of iterations compared to the SOR and GS method. Table 2 shows the decrement percentages number of iterations for FSAOR, HSAOR, FSSOR, HSSOR, HSGS methods compared to the FSGS method in solving the proposed example. Through the observation in Tables 1 and 2, found that applications of the half-sweep iteration idea reduce the execution time of the iterative method. In the meantime, decrement percentages of the execution time for FSAOR, HSAOR, FSSOR, HSSOR, HSG methods compared with FSGS method have been summarized in Table 2. In terms of accuracy, approximate solutions for the FSAOR and HSAOR methods are in good agreement compared to the FSSOR, HSAOR, FSGS and HSGS methods.

Generally, the numerical results prove that the HSAOR iterative method is a better method compared with the FSAOR, FSSOR, HSSOR, FSGS and HSGS methods in the sense of the complexity and execution time. This is due to the computational complexity of the HSAOR method is approximately 50% less than FSAOR and FSSOR methods.

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