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# Four Point Explicit Decoupled Group Iterative Method Applied to Two-Dimensional Helmholtz Equation

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**Abstract**

In this article, we consider the numerical solution of two-dimensional Helmholtz equation. The four point Explicit Decoupled Group (EDG) iterative method together with Gauss-Seidel (GS) is applied to solve a linear system generated from discretization of the finite difference scheme using the second order central difference. In addition, the formulation and implementation of the proposed method to solve the problem also presented. Numerical result and comparisons with other existing method are given to illustrate the efficiency of the proposed method.

**Keywords**: Helmholtz equation, Explicit Group Method, Explicit Decoupled Group Method, Gauss-Seidel Method, Finite Difference Scheme.

## 1 Introduction

From previous studies, many researchers have investigated several numerical methods such as finite difference, finite element, finite volume and boundary element methods to gain approximate solutions in solving any partial differential equations, which describes a certain problem in science and engineering [13,15]. In addition, the discovery on the half-sweep iterative method has been initiated by Abdullah [1]. However, the concept of this method is extension of the full-sweep iterative method, which is inspired by Evans [3] through Explicit Group iterative method to solve the two-dimensional Poisson equation. Following to that, further application of the full- and half-sweep iteration concepts have been extensively studied by many researchers; see Ibrahim and Abdullah [2]; Sulaiman et al. [7]; Akhir et al. [8,10,11]; Othman and Abdullah [14] and Yousif and Evans [17]. The basic idea of the half-sweep iterative methods is to reduce the computational complexities during iteration process, since the implementation of the half- sweep iterations will only consider nearly half of all interior node points in a solution domain respectively.

In this article, we study the effectiveness of using the four Point-EDGGS method by using second order finite difference approximate equation for solving problem (1). To show the capability of the four Point-EDGGS method, let us consider the following two-dimensional Helmholtz equation with dirichlet boundary conditions on $\Omega=\left[0,1\right]^2$.

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}-\alpha U &= f\left(x,y\right), \quad \left(x,y\right)\in\Omega \\ U\left(x,y\right) &= G\left(x,y\right), \quad \left(x,y\right)\in\Omega \end{aligned} \tag{1}$$

subject to the dirichlet boundary condition and statisfy the exact solution $U\left(x,y\right)=G\left(x,y\right),\ \left(x,y\right)\in D=\partial D$. Where $f\left(x,y\right)$ is given function with suffi-icient smoothness and $\alpha$ is the nonnegative constant.

The outline of this article is organized as follows. In Section 2, the formulation of the full- and half-sweep finite difference approximation equations will be elaborated. The latter section of this article will discuss the formulations and derivation of the four Point-EGGS as well as four Point-EDGGS, and some numerical results will be shown in fourth section to assert the performance of the proposed methods. Besides that, analysis on computational complexity is mentioned in Section 4. Meanwhile, conclusions and open problems are given in Section 5 and 6 respectively.

## 2 Second Order finite difference approximations equations

When Eq. (1) is solved by finite difference approximations equation, the most commonly used approximation is the standard full-sweep five points stencil can be written as,

$$U_{i-1,j}+U_{i+1,j}+U_{i,j-1}+U_{i,j+1}-\left(4+\alpha h^2\right)U_{i,j} = h^2 f_{i,j}, \tag{2}$$

Apart from Eq. (2), another type of approximation can be derived from the rotated finite difference approximation, which can be obtained by rotating *x*-*y* axis clockwise $45^o$ (Dahlquist and Bjorck, [16]). Thus the rotated finite difference approximation of Eq. (1) can be easily expressed as

$$U_{i+1,j+1} + U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} - \left(4 + 2\alpha h^2\right) U_{i,j} = 2h^2 f_{i,j}, \tag{3}$$

Now it can be clearly seen that application either Eq. (2) or (3) to each internal mesh point will result a large and sparse linear system where *A* and *f* are a square nonsingular matrix with a column matrix, respectively.

$$A\underset{\sim}{U} = \underset{\sim}{f}, \tag{4}$$

where *A* and *f* are a square nonsingular matrix with a column matrix, respectively. While $\underset{\sim}{U}$ is a column matrix. The solution of Eq. (4) can be obtained by direct or iterative method. Since the equation is large and sparse, the iterative method is suitable to solve this type of problem and it can be solved by point block iterative methods; see Abdullah [1].

## 3 Formulations of the GS Iterative Methods

As mentioned above, four Point-EGGS and four Point-EDGGS iterative methods will be applied to solve linear system generated form discretization of the problem (1), as shown in Eq.(4). Let the coefficient matrix, *A* be decomposed into form of

$$A = D - L - V \tag{5}$$

where *D*, *L* and *V* are the diagonals, negative lower triangulation and negative upper triangulation matrices, respectively. The methods for solving Eq. (1) can be classified into two categories which are direct and iterative methods. Gauss elimination and LU factorization are some examples of the direct methods to solve system of linear algebraic equations. Meanwhile, in this article we are focusing on iterative linear system solvers. According to Young [5,6], the usage of the iterative methods has the advantage since the matrix *A* is not distorted during the computation and the problem of the accumulation of rounding errors is less staid than direct methods. Based on Eq. (5), the general scheme for four Point-EGGS and four Point-EDGGS iterative methods can be written as

$$\underset{\sim}{U}^{(k+1)} = \left(D - L\right)^{-1}\left(V\underset{\sim}{U}^{(k)} + \underset{\sim}{f}\right) \tag{6}$$

Actually, the iterative methods attempt to find a solution to the system of linear equations by repeatedly solving the linear system using approximations to the vector $\underset{\sim}{U}$. Iterations for four Point-EGGS and four Point-EDGGS iterative methods continue until the solution is within a predetermined acceptable bound on the error.

## 3 Formulations of the Four point block iterative methods

In general, implementation of this method will be imposed onto solid node points in Figure 1 and 2 till the convergence test criterion will be met. Afterward approximate values of the remaining node points at the finite difference networks as shown in Figures 2 will be also calculated directly by using the same steps in the finite difference scheme; see, e.g., (Abdullah [1]; Ibrahim and Abdullah [2]; Sulaiman et al. [7]; Akhir et al. [8,10,11,12]; Othman and Abdullah [14]; Yousif and Evans [17]) and the references therein. For comparison purpose, this paper will also consider other two point iterative methods such as FSGS and HSGS iterative methods. Again these two point iterative methods basically will be formulated by using the corresponding finite difference approximation equations in Eq. (2) till (3).

### 3.1 Implementation of Four point-EGGS Iterative Method

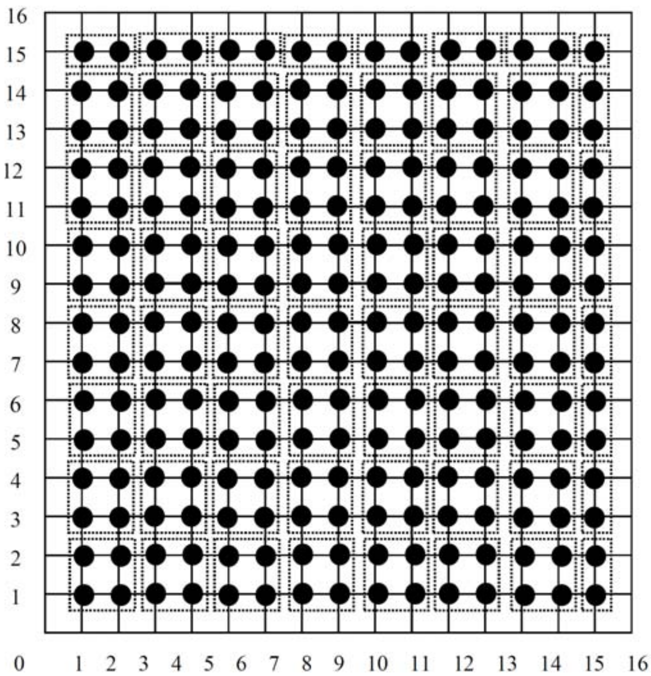


**Figure 1:** Implementation of the four point-EGGS iterative method at solution domain m=16.

For reason of formulation four Point-EDGGS iterative method, let consider a complete group of four points (4x4). By considering Eq. (1), this method can be generally expressed as

$$\begin{bmatrix} 4+h^2\alpha & -1 & 0 & -1 \\ -1 & 4+h^2\alpha & -1 & 0 \\ 0 & -1 & 4+h^2\alpha & -1 \\ -1 & 0 & -1 & 4+h^2\alpha \end{bmatrix} \begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i,j+1} \\ u_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}, \qquad (7)$$

where,

$$\begin{aligned} S_1 &= U_{i-1,j} + U_{i,j-1} - h^2 f_{i,j}, \\ S_2 &= U_{i+2,j} + U_{i+1,j-1} - h^2 f_{i+1,j}, \\ S_3 &= U_{i-1,j+1} + U_{i,j+2} - h^2 f_{i,j+1}, \\ S_4 &= U_{i+2,j+1} + U_{i+1,j+2} - h^2 f_{i+1,j+1}, \end{aligned}$$

Now by determining the inverse matrix of Eq. (7), the four point-EG method can be generally shown as

$$\begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i,j+1} \\ u_{i+1,j+1} \end{bmatrix}^{(k+1)} = \frac{1}{\beta} \begin{bmatrix} S_1 + S_a \\ S_2 + S_b \\ S_3 + S_b \\ S_4 + S_a \end{bmatrix}, \qquad (8)$$

where,

$$\begin{aligned} &\beta = \left(2+h^2\alpha\right)\left(4+h^2 r\right)^2 \left(6+h^2\alpha\right), \\ &a_1 = S_1 + S_4, \qquad a_2 = S_2 + S_3, \\ &S_a = \left(a_1 + 2a_2\right), \quad S_b = \left(2a_1 + a_2\right). \end{aligned}$$

Generally, the 4 Point-EGGS algorithm to solve problem (1) on $\Omega$ described as follows:

1. Divide the solution domain into one type as in Figure 1. Compute the values of $h^2$.
2. Iterate the intermediate solution $\underset{\sim}{U}$ of point type ● using Eq. (3)

$$U_{i-1,j} + U_{i+1,j} + U_{i,-1} + U_{i,j+1} - \left(4+\alpha h^2\right)U_{i,j} = h^2 f_{i,j}$$

3. Check the convergence. Otherwise repeat the iteration cycle (i.e., go to step 2)
4. Stop

Further details of the method can be found in (Abdullah [1]; Evans [3]; Akhir et al. [9])

## 3.2 Implementation of Four Point-EDGGS Iterative Method

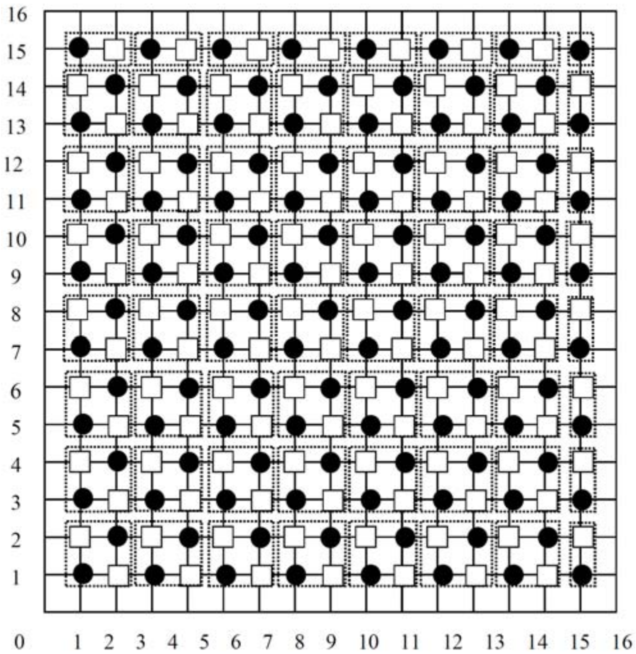


**Figure 1:** Implementation of the four point-EDGGS iterative method at solution domain m=16.

Let assume that the solution at any group of four points, (4x4). From Eq. (1), this method can be expressed in the following system of linear algebraic equations

$$\begin{bmatrix} 4+2h^2\alpha & -1 & 0 & 0 \\ -1 & 4+2h^2\alpha & 0 & 0 \\ 0 & 0 & 4+2h^2\alpha & -1 \\ 0 & 0 & -1 & 4+2h^2\alpha \end{bmatrix} \begin{bmatrix} u_{i,j} \\ u_{i+1,j} \\ u_{i,j+1} \\ u_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}, \tag{9}$$

where,

$$S_1 = U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} - 2h^2 f_{i,j},$$
$$S_2 = U_{i,j+2} + U_{i+2,j} + U_{i+2,j+2} - 2h^2 f_{i+1,j+1},$$
$$S_3 = U_{i,j-1} + U_{i+2,j-1} + U_{i+2,j+1} - 2h^2 f_{i+1,j},$$
$$S_4 = U_{i-1,j} + U_{i-1,j+2} + U_{i+1,j+2} - 2h^2 f_{i,j+1}.$$

By splitting Eq. (6) this linear system can be written to a decoupled group of (2×2) linear systems, which are independently for each other. The four point-EDGGS iterative method can be easily shown as

$$\begin{bmatrix} u_{i,j} \\ u_{i+1,j+1} \end{bmatrix}^{(k+1)} = \frac{1}{\beta}\begin{bmatrix} 4+2h^2\alpha & 1 \\ 1 & 4+2h^2\alpha \end{bmatrix}\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \tag{10}$$

$$\begin{bmatrix} u_{i+1,j} \\ u_{i,j+1} \end{bmatrix}^{(k+1)} = \frac{1}{\beta}\begin{bmatrix} 4+2h^2\alpha & 1 \\ 1 & 4+2h^2\alpha \end{bmatrix}\begin{bmatrix} S_3 \\ S_4 \end{bmatrix}, \tag{11}$$

where

$$\beta = \left(4+2\alpha h^2\right)^2 - 1.$$

In this method, the $\Omega$ is divided into two types of points (i.e. ● and ○) as shown in Figure 2. The solutions on any group of points (either ● or ○) can only be implemented by only involving the same type of point. The four point-EDGGS algorithm may be described as follows

1. Divide the solution domain into two types as in Figure 2. Compute the values of $2h^2$.
2. Iterate the intermediate solution $\underset{\sim}{U}$ of point type ● using Eq. (3)

   $$U_{i+1,j+1} + U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} - \left(4+2\alpha h^2\right)U_{i,j} = 2h^2 f_{i,j}$$

3. Check the convergence. If converge evaluate the rest of points (i.e., ○) using,

   3.1 $U_{i-1,j} + U_{i+1,j} + U_{i,-1} + U_{i,j+1} - \left(4+\alpha h^2\right)U_{i,j} = h^2 f_{i,j}$

   respectively. Otherwise repeat the iteration cycle (i.e., go to step 2)
4. Stop

The method was introduced by Abdullah [1] and the details of the method can be found (Ibrahim and Abdullah [2]; Akhir et al. [10, 12]; Yousif and Evans [17])

## 4 Numerical Simulations

In order to verify the effectiveness of the proposed methods, several numerical tests were carried out on the following two-dimensional Helmholtz equations problem. In comparison, the Full-Sweep Gauss-Seidel (FSGS) method acts as the control of comparison of numerical result. Three criteria such as number of iterations, execution time and maximum absolute error will be considered in comparison for FSGS. In the following examples, the convergence test for the implementation of the iterative methods considered the tolerance error, $\varepsilon = 10^{-10}$.

Example 1 (Evans [3])

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \alpha U = 6 - \alpha\left(2x^2 + y^2\right), \quad (x,y) \in D = [0,1] \times [0,1].$$

with the exact solution

$$U(x,y) = 2x^2 + y^2.$$

Example 2 (Evans et al. [4])

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - 2U = \left(12x^2 + 3x^4\right)\sin y, \quad (x,y) \in D = [0,1] \times [0,2\pi].$$

with the exact solution

$$U(x,y) = x^4 \sin y.$$

Result of numerical experiments, which were obtained from implementations of the FSGS, HSGS, four Point-EGGS and four point EDGGS methods for Examples 1 and 2, has been recorded in Tables 2-3.

## 5 Computational Complexity Analysis

The computational effort measured by number of computer operations needed to obtained a (sufficient accurate) solution by the three methods discussed for solving problem (1) can be estimated. Assume the solution domain is large with $m^2$ number of internal mesh points with $m = n - 1$. In their iterative process, the four Point-EGGS and four point-EDGGS methods requires $(m-1)^2$ and $(m-1)^2/2$ internal mesh points respectively.

Note that our estimate on this computational complexity is based on the arithmetic operations performed per iteration and execution time for the additions/subtraction (ADD/SUB) and multiplications/divisions (MUL/DIV) operations. Hence the number of operations of operations required (excluding convergence test and direct solution) for four point-EGGS and four point-EDGGS methods as described in Section 3 are respectively given as follows in Table 1,

**Table 1**: Total number of arithmetic operations per iteration for four point-EGGS and four point-EDGGS methods.

| **Methods** | **Iteration** | | **Direct** | |
|---|---|---|---|---|
| | **ADD/SUB** | **MUL/DIV** | **ADD/SUB** | **MUL/DIV** |
| **4-EG** | $8m^2k+4(2m-1)$ | $4m^2k+(2m-1)$ | - | - |
| **4-EDG** | $2m^2k+2m$ | $2m^2k+2m$ | $2m^2$ | $m^2/2$ |

Note: k is the number of iterations and $m^2$ represents $(m-1)$

## 6 Conclusion

In the previous section, we present formulation of full-, half, and quarter-sweep approximation equations based on the second orders finite difference method can easily generate a system of linear algebraic equations as shown in Eq. (4). From Tables 2 and 3, clearly show that by applying half-sweep approach can reduce the number of iterations compared to FSGS method. Table 4 shows decrement percentages number of iterations for four point-EGGS and four point-EDGGS methods. Through the surveillance in Tables 2 and 3, we found that application of the half-sweep concepts reduces the execution time of the iterative method. Meanwhile, decrement percentages of the execution time for HSGS, four point-EGGS and four point EDGGS methods compared with FSGS method have been summarized in Table 4. In addition, the accuracy approximate solutions for four point-EGGS and four point-EDGGS methods are in good agreement compared with the FSGS method. For future works, this study will be continued to investigate on the use and the development of Modified SOR iterative method as an alternative approach to speed up the execution time for solving two-dimensional Helmholtz equation (Akhir et. al [9,10]).

**Table 2**: Comparison of a Number of Iterations, Execution Times (Seconds) and Maximum Absolute Error for the Iterative Methods (Example1) at $\alpha = 10$.

| Mesh size | Method | Numbers of Iterations | Execution Times (Seconds) | Maximum Absolute Error |
|---|---|---|---|---|
| 32 | **FSGS** | 1326 | 0.26 | 6.8176e-10 |
| | **4-EGGS** | 640 | 0.21 | 3.3181e-10 |
| | **4-EDGGS** | 354 | 0.09 | 1.60718e-9 |
| 64 | **FSGS** | 4910 | 1.38 | 2.7407e-10 |
| | **4-EGGS** | 2553 | 0.77 | 5.5044e-10 |
| | **4-EDGGS** | 1321 | 0.35 | 6.8224e-9 |
| 128 | **FSGS** | 18085 | 15.43 | 1.1004e-10 |
| | **4-EGGS** | 9426 | 7.44 | 1.3653e-10 |
| | **4-EDGGS** | 4902 | 3.74 | 2.7410e-9 |
| 256 | **FSGS** | 66177 | 210.11 | 4.4067e-10 |
| | **4-EGGS** | 34618 | 107.09 | 2.2029e-10 |
| | **4-EDGGS** | 18071 | 56.48 | 1.1004e-9 |

**Table 3**: Comparison of a Number of Iterations, Execution Times (Seconds) and Maximum Absolute Error for the Iterative Methods (Example 2).

| Mesh size | Method | Numbers Of Iterations | Execution Times (Seconds) | Maximum Absolute Error |
|---|---|---|---|---|
| 32 | **FSGS** | 1694 | 0.34 | 9.7880e-3 |
| | **4-EGGS** | 886 | 0.28 | 9.7880e-3 |
| | **4-EDGGS** | 458 | 0.15 | 9.7889e-2 |
| 64 | **FSGS** | 6175 | 1.56 | 9.7889e-3 |
| | **4-EGGS** | 3235 | 1.01 | 9.7889e-3 |
| | **4-EDGGS** | 1686 | 0.52 | 9.7888e-2 |
| 128 | **FSGS** | 22340 | 16.8 | 9.7906e-3 |
| | **4-EGGS** | 11848 | 9.11 | 9.7906e-3 |
| | **4-EDGGS** | 6159 | 5.05 | 9.7888e-2 |
| 256 | **FSGS** | 80028 | 225.82 | 9.7923e-3 |
| | **4-EGGS** | 42318 | 124.43 | 9.7922e-3 |
| | **4-EDGGS** | 23307 | 62.71 | 9.7922e-2 |

**Table 4**: Decrement percentages of the number of iterations and execution time for four point-EGGS and four point-EDGGS methods compared with the FSGS method.

| Example | Methods | Numbers of Iterations (%) | Execution Time (%) |
|---|---|---|---|
| **1** | **4-EGGS** | 47.69 - 51.73 | 19.23 - 51.78 |
| | **4-EDGGS** | 72.69 - 73.30 | 65.38 - 75.76 |
| **2** | **4-EGGS** | 46.97 - 47.70 | 17.64 - 45.77 |
| | **4-EDGGS** | 70.88 - 72.96 | 55.88 - 72.23 |

## References

[1] A.R. Abdullah, The Four Point Explicit Decoupled Group (EDG) Method: A Fast Poisson Solver, Int. J. Comp. Math., **38** (1991), 61-70.

[2] A. Ibrahim, and A.R. Abdullah, Solving the two-dimensional diffusion equation by the four point explicit decoupled group (EDG) iterative method. Int. J. Comp. Math, **58** (1995), 253-256.

[3] D.J. Evans, Group Explicit Iterative methods for solving large linear systems", Int. J. Comp. Math, **17** (1985), 81-108.

[4] D.J. Evans, M. Ergut and H. Bulut, The numerical solution of multidimensional partial differential equations by decomposition method, Int. J. Comp. Math, **80** (2003), 1189-1198.

[5] D.M. Young, Iterative solution of large linear systems. London, Academic Press, 1971.

[6] D.M. Young, Second-degree iterative methods for the solution of large linear systems. J. Approx. The, **5** (1972)., 37-148.

[7] J. Sulaiman, M.K. Hasan and M. Othman, The Half-Sweep Iterative Alternating Decomposition Explicit (HSIADE) method for diffusion equations, In Computational and Information Science, J. Zhang, J.H. He and Y.Fu (Eds). Berlin: Springer-Verlag, (2004), 57-63.

[8] M.K. Akhir, M. Othman, J. Sulaiman and Z.A Majid, A New Half-Sweep Modified SOR for Solving Two-Dimensional Helmholtz Equation. Proceedings of International Conference on Intelligent Network and Computing (ICNIC), 26-28 November (2010), 416-419.

[9] M.K. Akhir, M. Othman, J. Sulaiman, Z.A Majid, and M. Suleiman, Numerical Solutions of Helmholtz Using A New Four Point EGMSOR Iterative Method equations, Appl. Math. Science, **80** (2011), 3991-4004.

[10] M.K. Akhir, M.Othman, J. Sulaiman, Z.A Majid, and M. Suleiman, The Four Point-EDGMSOR Iterative Method for Solution of 2D Helmholtz Equations, In Communications in Computer and Information Science, A. Abd Manaf. Berlin: Springer-Verlag, **253** (2011), 218-227.

[11] M.K. Akhir, M. Othman, J, Sulaiman, and Z.A. Majid and M, Suleiman, Half Sweep Iterative Method for Solving Two-Dimensional Helmholtz Equations, Int. J. Appl. Math. Stat, 29 (2012), 101-109.

[12] M.K. Akhir, M. Othman, J, Sulaiman, and M, Suleiman, A New Octo Modified Explicit Group Iterative Method for the Solution of 2D Elliptic PDEs, Appl. Math. Science, **6** (2012), 1505-1524.

[13] M. Nabavi, M.H.K Sidiqui, and J. Daraghi, A new 9-point sixth-order accurate compact finite difference methods for Helmholtz equation, J. Sound Vibrat., **307** (2007), 972-982.

[14] M. Othman and A.R. Abdullah, The Halfsweeps Multigrid Method as a Fast Multigrid Poisson Solver. Int. J. Comp. Math, **69** (1998)., 219–229.

[15] T. Belytschko, Y. Kongraus, D. Organ, D. Fleming, and P. Krystal P (2007). Meshless methods: An overview and recent developments, J. Comput. Meth. Appl., Mecha. Eng, **139** (2007), 3–47.

[16] V. Dahlquist and A. Bjorck, Numerical Method, New York, Prentice Hall 1974.

[17] W.S. Yousif, and D.J. Evans, Explicit De-coupled Group Iterative Methods and Their Implementations, Int. J. Paral. Emer. Dsitri. Sys, **7** (1995), 53-71.

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