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Coefficient Estimates for Certain Subclasses of Bi-Univalent Functions

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Abstract

Let Σ denote the class of bi-univalent functions in $D = \{z \in \mathbb{C} : |z| < 1\}$. In this paper, we consider two subclasses of Σ defined in the open unit disk D which are denoted by $S^*_{s,\Sigma}(\phi)$ and $C_{s,\Sigma}(\phi)$. Besides, we find upper bounds for the second and third coefficients for functions in these subclasses.

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1 Introduction

Let A denote the class of functions $f(z)$ normalized by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in D \tag{1}$$

which are analytic in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Further, let S denote the subclass of functions in A which are univalent in D. Some of the important and well-investigated subclasses of S include the class of starlike functions and the class of convex functions which are denoted by S^* and C respectively. By definition, we have

$$S^* = \left\{ f : f \in A \;\; and \;\; Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \;\; z \in D \right\} \tag{2}$$

and

$$C = \left\{ f : f \in A \;\; and \;\; Re\left(1 + \frac{zf''(z)}{f'(z)}\right) 0, \;\; z \in D \right\} \tag{3}$$

It readily follows from definitions (2) and (3) that

$$f(z) \in C \Longleftrightarrow zf'(z) \in S^*. \tag{4}$$

The Koebe one-quarter theorem [4] states that the image of D under every function $f(z)$ from S contains a disk of radius $\frac{1}{4}$. Thus every function $f(z) \in S$ has an inverse $f^{-1}(f(z))$ defined by $f^{-1}(f(z)) = z \;\; (z \in D)$ and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function $f^{-1}(w)$ is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{5}$$

A function $f(z) \in A$ is said to be bi-univalent in D if both $f(z)$ and $f^{-1}(w)$ are univalent in D. Let Σ denote the class of bi-univalent functions given by the Taylor-Maclaurin series expansion (1). Some examples of function in the class Σ are $\frac{z}{1-z}, -\log(1-z)$ and $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$. However, the familiar Koebe function is not a member of Σ. Other examples of function in S such as $z - \frac{z^2}{2}$ and $\frac{1}{1-z^2}$ are also not members of Σ.

Lewin [5] investigated the class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Netanyahu [7], on the other hand, showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. Brannan and Taha [2] introduced certain subclasses of Σ similar to the familiar subclasses of S consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and obtained estimates on the initial coefficients. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \ \ (n \in N \backslash \{1, 2\}; N := 1, 2, 3, \ldots)$$

is still an open problem.

If the functions $f(z)$ and $g(z)$ are analytic in D then $f(z)$ is said to be subordinate to $g(z)$ written as $f(z) \prec g(z), (z \in D)$ if there exists a Schwarz function $w(z)$, analytic in D, with $w(0) = 0, |w(z)| < 1, (z \in D)$ such that $f(z) = g(w(z)), (z \in D)$.

In [6], the authors introduced the class $S^*(\phi)$ of Ma-Minda starlike functions and the class $C(\phi)$ of Ma-Minda convex functions, unifying previously studied classes related to starlike and convex functions. The class $S^*(\phi)$ consists of all the functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$ whereas $C(\phi)$ is formed with functions $f \in A$ for which the subordination $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$ holds. The function ϕ is analytic and univalent function with positive real part in D with $\phi(0) = 0, \phi'(0) > 0$ and ϕ maps the unit disk D onto a region starlike with respect to 1 and symmetric with respect to the real axis. Taylor's series expansion of such function is of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots \tag{6}$$

where all coefficients are real and $B_1 > 0$.

In [10], Sakaguchi introduced the class S_s^* of starlike functions with respect to symmetric points in D, consisting of functions $f \in A$ that satisfy the condition

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in D$$

and in [3], Das and Singh introduced the class C_s of convex functions with respect to symmetric points in D, consisting of functions $f \in A$ that satisfy the condition

$$Re\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0, \quad z \in D.$$

Motivated by the earlier works of [10], [3] and [6] and considering functions $f \in \Sigma$, this paper introduce two subclasses of Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses.

2 Preliminary Result and Definitions

In order to derive our main results, we need the following lemma.

Lemma 2.1. *([9]) If $p(z) \in P$ then $|p_k| \leq 2$ for each k, where P is the family of all functions $p(z)$ analytic in D for which $Re(p(z)) > 0$, $p(z) = 1 + p_1 z + p_2 z^2 + ...$ for $z \in D$.*

Definition 2.1. *A function $f(z) \in \Sigma$ is said to be in class $S^*_{s,\Sigma}(\phi)$ if the following subordinations hold:*

$$\frac{zf'(z)}{f(z) - f(-z)} \prec \phi(z) \tag{7}$$

and

$$\frac{wg'(w)}{g(w) - g(-w)} \prec \phi(w) \tag{8}$$

where $g(w) = f^{-1}(w)$ is given by (5).

Definition 2.2. *A function $f(z) \in \Sigma$ is said to be in class $C_{s,\Sigma}(\phi)$ if the following subordinations hold:*

$$\frac{(zf'(z))'}{(f(z) - f(-z))'} \prec \phi(z) \tag{9}$$

and

$$\frac{(wg'(w))'}{(g(w) - g(-w))'} \prec \phi(w) \tag{10}$$

where $g(w) = f^{-1}(w)$ is given by (5).

3 Main Results

For functions in the class $S^*_{s,\Sigma}(\phi)$, the following result is obtained.

Theorem 3.1. *If $f \in S^*_{s,\Sigma}(\phi)$ is given by (1) then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2\left|B_1^2 + 2(B_1 - B_2)\right|}} \tag{11}$$

and

$$|a_3| \leq \frac{1}{2}B_1\left(1 + \frac{1}{2}B_1\right). \tag{12}$$

Proof. Let $f \in S^*_{s,\Sigma}(\phi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : D \to D$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z) - f(-z)} = \phi(u(z)) \tag{13}$$

and

$$\frac{wg'(w)}{g(w) - g(-w)} = \phi(v(w)). \tag{14}$$

Define the functions r_1 and r_2 by

$$r_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + ...$$

and

$$r_2(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + b_1 z + b_2 z^2 + ...$$

or equivalently

$$u(z) = \frac{r_1(z) - 1}{r_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + ...\right) \tag{15}$$

and

$$v(z) = \frac{r_2(z) - 1}{r_2(z) + 1} = \frac{1}{2}\left(b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + ...\right). \tag{16}$$

Then r_1 and r_2 are analytic in D with $r_1(0) = 1 = r_2(0)$. Since $u, v : D \to D$, the functions r_1 and r_2 have a positive real part in D and $|b_i| \leq 2$ and $|c_i| \leq 2$. In view of (13)-(16), clearly

$$\frac{zf'(z)}{f(z) - f(-z)} = \phi\left(\frac{r_1(z) - 1}{r_1(z) + 1}\right) \tag{17}$$

and

$$\frac{wg'(w)}{g(w) - g(-w)} = \phi\left(\frac{r_2(w) - 1}{r_2(w) + 1}\right). \tag{18}$$

Using (15) and (16) together with (6), it is evident that

$$\phi\left(\frac{r_1(z) - 1}{r_1(z) + 1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + ... \tag{19}$$

and

$$\phi\left(\frac{r_2(w) - 1}{r_2(w) + 1}\right) = 1 + \frac{1}{2}B_1 b_1 w + \left(\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right) w^2 + ... \tag{20}$$

Since $f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

Since

$$\frac{zf'(z)}{f(z) - f(-z)} = 1 + 2a_2 z + 2a_3 z^2 + ...$$

and

$$\frac{wg'(w)}{g(w) - g(-w)} = 1 - 2a_2 w + 2(2a_2^2 - a_3)w^2 + ...$$

it follows from (17)-(20) that

$$2a_2 = \frac{1}{2} B_1 c_1 \tag{21}$$

$$2a_3 = \frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \tag{22}$$

$$-2a_2 = \frac{1}{2} B_1 b_1 \tag{23}$$

and

$$2(2a_2^2 - a_3) = \frac{1}{2} B_1 \left(b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2 \tag{24}$$

From (21) and (23), it follows that

$$c_1 = -b_1. \tag{25}$$

Now (21)-(25) yield

$$a_2^2 = \frac{B_1^3 (b_2 + c_2)}{8 (B_1^2 + 2 (B_1 - B_2))}$$

which, in view of the inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives us the estimate on $|a_2|$ as asserted in (11).

By subtracting (24) from (22), further computation using (21) and (25) lead to

$$a_3 = \frac{B_1^2 (c_1^2 + b_1^2)}{32} + \frac{B_1 (c_2 - b_2)}{8}$$

and this yields the estimate given in (12). The proof of Theorem 3.1 is completed.

The result in Theorem 3.1 is similar to Theorem 2.3 in [8] if $\alpha = 0$.

By using the similar approach as Theorem 3.1, we obtain the following result for functions $f \in C_{s,\Sigma}(\phi)$.

Theorem 3.2. *If $f \in C_{s,\Sigma}(\phi)$ is given by (1) then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{2\left|3B_1^2+8(B_1-B_2)\right|}} \tag{26}$$

and

$$|a_3| \leq \frac{1}{2}B_1\left(\frac{1}{3}+\frac{1}{8}B_1\right). \tag{27}$$

Proof. Let $f \in C_{s,\Sigma}(\phi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : D$, with $u(0) = v(0) = 0$, satisfying

$$\frac{(zf'(z))'}{(f(z)-f(-z))'} = \phi(u(z)) \tag{28}$$

and

$$\frac{(wg'(w))'}{(g(w)-g(-w))'} = \phi(v(w)). \tag{29}$$

Since

$$\frac{(zf'(z))'}{(f(z)-f(-z)')} = 1 + 4a_2z + 6a_3z^2 + ...$$

and

$$\frac{(wg'(w))'}{(g(w)-g(-w))'} = 1 - 4a_2w + 6(2a_2^2 - a_3)w^2 + ...$$

it follows from (19), (20), (28) and (29)that

$$4a_2 = \frac{1}{2}B_1c_1 \tag{30}$$

$$6a_3 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2 \tag{31}$$

$$-4a_2 = \frac{1}{2}B_1b_1 \tag{32}$$

and

$$6(2a_2^2 - a_3) = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2 \tag{33}$$

From (30) and (32), it follows that

$$c_1 = -b_1. \tag{34}$$

Equations (30)-(34) yield

$$a_2^2 = \frac{B_1^3\left(b_2+c_2\right)}{8\left(3B_1^2+8\left(B_1-B_2\right)\right)}$$

which, in view of the inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives the estimate on $|a_2|$ as asserted in (26).

Further computation using (30)-(34) lead to

$$a_3 = \frac{B_1^2\left(b_1^2 + c_1^2\right)}{128} + \frac{B_1\left(c_2 - b_2\right)}{24}$$

and this yields the estimate given in (27). The proof of Theorem 3.2 is completed.

The result in Theorem 3.2 is similar to Theorem 2.3 in [8] if $\alpha = 1$.

For functions in the class $S^*_{s,\Sigma}(\phi)$, we obtained the result on Fekete-Szegö inequalities as follows.

Theorem 3.3. *Let f given by (1) be in the class $S^*_{s,\Sigma}(\phi)$ and $\mu \in \Re$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_1}{2}, & |\mu - 1| \leq \left|1 + 2\left(\frac{B_1 - B_2}{B_1^2}\right)\right| \\ \\ \frac{|1-\mu|B_1^3}{2\left|B_1^2 + 2(B_1 - B_2)\right|}, & |\mu - 1| \geq \left|1 + 2\left(\frac{B_1 - B_2}{B_1^2}\right)\right| \end{cases}$$

Finally, we give the result on Fekete-Szegö inequalities for functions in the class $C_{s,\Sigma}(\phi)$.

Theorem 3.4. *Let f given by (1) be in the class $C_{s,\Sigma}(\phi)$ and $\mu \in \Re$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_1}{6}, & |\mu - 1| \leq \frac{1}{3}\left|3 + 8\left(\frac{B_1 - B_2}{B_1^2}\right)\right| \\ \\ \frac{|1-\mu|B_1^3}{2\left|3B_1^2 + 8(B_1 - B_2)\right|}, & |\mu - 1| \geq \frac{1}{3}\left|3 + 8\left(\frac{B_1 - B_2}{B_1^2}\right)\right| \end{cases}$$

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