COEFFICIENT INEQUALITIES OF SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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ABSTRACT

In this thesis, a class of functions $f(z)$ which are analytic in the open unit disk $\mathcal{D} := \{z: |z| < 1\}$ is denoted by $\mathcal{A}$. Next, $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions and normalized by $f(0) = 0$ and $f'(0) = 1$. The main subclasses of $\mathcal{S}$ are the classes of starlike, convex, close-to-convex and quasi-convex functions which can be represented as $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{Q}$ respectively. Every univalent function f has an inverse function defined by $f^{-1}(f(z)) = z$ and $f\left(f^{-1}(w)\right) = w$ where $|w| < r_0(f)$ and $r_0(f) \geq \frac{1}{4}$ with $f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$. A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{D}$ if both f and f^{-1} are univalent in $\mathcal{D}$. Further, σ denoted the class of bi-univalent functions in $\mathcal{D}$. Hence, in this thesis, new subclasses of σ were proposed by considering functions $f \in \sigma$ and the coefficient estimates for these classes are obtained. Furthermore, the upper bounds of the Fekete-Szegö inequalities and second Hankel determinant are obtained for certain subclasses of σ.

ABSTRAK

SUBKELAS BAGI FUNGSI BI-UNIVALEN

Di dalam tesis ini, kelas fungsi $f(z)$ *yang analisis di dalam cakera unit terbuka* $\mathcal{D} := \{z: |z| < 1\}$ *ditandakan sebagai* $\mathcal{A}$. *Seterusnya,* $\mathcal{S}$ *melambangkan subkelas bagi* $\mathcal{A}$ *yang mengandungi fungsi univalen dan ternormal dengan* $f(0) = 0$ *dan* $f'(0) = 1$. *Subkelas utama bagi* $\mathcal{S}$ *ialah kelas fungsi bakbintang, cembung, hampir cembung dan kuasi cembung yang masing-masing diwakilkan sebagai* $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ *dan* $\mathcal{Q}$. *Setiap fungsi* f *mempunyai fungsi songsangan yang ditakrifkan sebagai* $f^{-1}(f(z)) = z$ *dan* $f\left(f^{-1}(w)\right) = w$ *dengan* $|w| < r_0(f)$ *dan* $r_0(f) \geq \frac{1}{4}$ *bagi* $f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$. *Suatu fungsi* $f \in \mathcal{A}$ *dikatakan bi-univalen di dalam* $\mathcal{D}$ *jika kedua-dua fungsi* f *dan* f^{-1} *adalah univalen di dalam* $\mathcal{D}$. *Selanjutnya,* σ *ditandakan sebagai kelas fungsi bi-univalen di dalam* $\mathcal{D}$. *Justeru itu, di dalam tesis ini, subkelas baru bagi* σ *diperkenalkan dengan mempertimbangkan fungsi* $f \in \sigma$ *dan anggaran pekali bagi kelas-kelas tersebut diperoleh. Di samping itu, batasan atas bagi ketaksamaan Fekete-Szegö dan penentu Hankel ke-dua juga diperoleh untuk suatu subkelas bagi* σ.

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LIST OF SYMBOLS

$\leq$	-	less than or equal to
$<$	-	less than
$\geq$	-	greater than or equal to
$>$	-	greater than
$\lvert\ \rvert$	-	modulus
$\in$	-	an element of
σ	-	class of bi-univalent functions
∞	-	infinity
Re	-	real part of
$\subset$	-	subset of
$\mathbb{C}$	-	set of complex numbers
$\mathcal{D}$	-	unit disk
$\mathcal{X}$	-	closed square
E	-	domain
arg	-	argument
max	-	maximum value
$\mathcal{P}$	-	set of functions with positive real part
$\mathcal{A}$	-	class of analytic functions
$\mathcal{S}$	-	class of univalent functions
$\mathcal{S}^*$	-	class of starlike functions
$\mathcal{C}$	-	class of convex functions
K	-	class of close-to-convex functions
Q	-	class of quasi-convex functions

$\mathcal{S}_s^*(\alpha)$	-	class of starlike functions with respect to symmetric points of order α
$\mathcal{C}_s(\alpha)$	-	class of convex functions with respect to symmetric points of order α
$\mathcal{S}^*(\alpha)$	-	class of starlike functions of order α
$\mathcal{C}(\alpha)$	-	class of convex functions of order α
$\mathcal{S}_\sigma^*(\alpha)$	-	class of bi-starlike functions of order α
$\mathcal{C}_\sigma^*(\alpha)$	-	class of bi-convex functions of order α
$\mathcal{SS}^*(\alpha)$	-	class of strongly starlike functions of order α
$\mathcal{SC}^*(\alpha)$	-	class of strongly convex functions of order α
$\mathcal{ST}_\sigma(\varphi)$	-	class of bi-starlike functions of Ma-Minda type
$\mathcal{CV}_\sigma(\varphi)$	-	class of bi-convex functions of Ma-Minda type
$f(z)$	-	function of z
$f^{-1}(f(z))$	-	inverse of $f(z)$
$k(z)$	-	Koebe function
$L_0(z)$	-	Möbius function
$H_q(n)$	-	q-th Hankel determinant

$\mathcal{S}_s^*(\alpha)$	-	class of starlike functions with respect to symmetric points of order α
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$\mathcal{S}_\sigma^*(\alpha)$	-	class of bi-starlike functions of order α
$\mathcal{C}_\sigma^*(\alpha)$	-	class of bi-convex functions of order α
$\mathcal{SS}^*(\alpha)$	-	class of strongly starlike functions of order α
$\mathcal{SC}^*(\alpha)$	-	class of strongly convex functions of order α
$\mathcal{ST}_\sigma(\varphi)$	-	class of bi-starlike functions of Ma-Minda type
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LIST OF APPENDIXES

CHAPTER 1

PRELIMINARIES

1.1 Introduction

One of the aspects of the theory of analytic functions of a complex variable is the study of geometric function theory. This study was founded around the turn of the 20th century and still remains as one of the most active fields researched by others scholars. Geometric function theory studies the geometric properties of analytic functions and is mainly concerned with the power series of the form

$$f(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots$$

in a complex variable z that is convergent in a domain E. Such a power series can be interpreted as a mapping of the domain E in the z-plane onto the same range set F in the w-plane.

This thesis considers $\mathcal{A}$ to be the class of analytic functions in the open unit disk $\mathcal{D} = \{z: |z| < 1\}$. According to Spiegel (1964), if the derivative $f'(z)$ exists at all points z of a regime E, then $f(z)$ is said to be analytic in E and is referred to as an analytic function in E or a function analytic in E. The terms regular and holomorphic are sometimes used as synonyms for analytic. A function $f(z)$ is said to be analytic at a point z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists. Let $f(z) = {}^1/_z$, $g(z) = z^{1/2}$ and $h(z) = e^z$ for $|z - i| < 1$. All three functions are analytic in this region and in particular are analytic in i.

1.2 Univalent Functions

From the geometric point of view, univalent functions are the simplest analytic functions. Various other terms are used for this concept. Thus, univalent functions are called simple or schlicht (the German word for simple). The Russians refer to such functions as ornolistni, which means single-sheeted. According to Ahuja (1986), a function $f(z)$ that is analytic in E is said to be univalent in E, if it never takes the same value twice, that is, $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$, where $z_1, z_2 \in E$. Certain simplifying assumptions are necessary in the theory of univalent functions. The first assumption is to take the unit disk $\mathcal{D} = \{z : |z| < 1\}$ in place of the arbitrary domain E. The second assumption is to take the normalization conditions which are $f(0) = 0$ and $f'(0) = 1$. With these assumptions, $f(z)$ can be rewritten in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}, \quad z \in \mathcal{D} \tag{1.1}$$

Further, let

$$\mathcal{S} = \{f : f \in \mathcal{A} \text{ and } f \text{ is univalent in } \mathcal{D}\}$$

An example of a univalent function is the function $g(z) = \frac{1+z}{1-z}$. Another example of univalent function is the Koebe function in the form

$$k(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right) - 1\right] = \frac{z}{(1-z)^2}, \qquad z \neq 1$$

that maps $\mathcal{D}$ onto the entire complex plane except the slit along the negative real axis from $-\infty$ to $-\frac{1}{4}$.

1.3 Subclasses of Univalent Functions

We let $\mathcal{S}$ be the subclasses of $\mathcal{A}$ in $\mathcal{D}$. Some of the most important subclasses of $\mathcal{S}$ are given as follow.

Definition 1.1 (Goodman, 1975) A set E in the plane is said to be starlike with respect to w_0 an interior point of E if each ray with initial point w_0 intersects the interior of E in a set that is either a line segment or a ray. If a function $f(z)$ maps $\mathcal{D}$ onto a domain that is starlike with respect to w_0, then we say that $f(z)$ is starlike with respect to w_0. In the special case that $w_0 = 0$, we say that $f(z)$ is a starlike function.

The class of all starlike functions in $\mathcal{D}$ is denoted by $\mathcal{S}^*$. It was first studied by Alexander in 1916. The most common example of starlike function is the Koebe function, $k(z) = \frac{z}{(1-z)^2}$ because it maps $\mathcal{D}$ onto the entire complex plane excluding the slit $-\infty < w \leq -\frac{1}{4}$. Robertson (1936) showed that $f \in \mathcal{S}^*$ if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathcal{D}$$

Another important subclass of $\mathcal{S}$ consist of the convex functions.

Definition 1.2 (Goodman, 1975) A set E in the plane is said to be convex if for every pair of points w_1 and w_2 in the interior of E, the line segment joining w_1 and w_2 is also in the interior of E. If a function $f(z)$ maps $\mathcal{D}$ onto a convex domain, then $f(z)$ is called a convex function.

The class of all convex functions in $\mathcal{D}$ is denoted by $\mathcal{C}$. It was first studied in 1931 by Jensen. The Mobius function, $L_0(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n$, is one example of convex function because it maps $\mathcal{D}$ onto a half plane. Robertson (1936) observed that function $f \in S$ is convex in $\mathcal{D}$ if and only if

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\quad z\in\mathcal{D}$$

In 1952, Kaplan introduced the class K of all close-to-convex functions in $\mathcal{D}$.

Definition 1.3 (Kaplan, 1952) Let $f(z)$ be analytic for $|z|<R$. Then $f(z)$ is close-to-convex for $|z|<R$ if there exists a function $\phi(z)$, convex and schlicht for $|z|<R$, such that $\frac{f'(z)}{\phi'(z)}$ has positive real part for $|z|<R$.

It will be convenient to exclude reference to the circular domain of definition when $R=1$. Hence, a close-to-convex function will mean a function which is close-to-convex for $|z|<1$.

The classes of $\mathcal{S}^*$ and $\mathcal{C}$ can be generalized to the class of the starlike and convex functions of order α as follows.

Definition 1.4 (Goodman, 1975) A function $f(z)$ in the form (1.1) is said to be starlike of order α in $\mathcal{D}$ if for all $z\in\mathcal{D}$

$$Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\quad z\in\mathcal{D}$$

where $0\le\alpha<1$. We denote $\mathcal{S}^*(\alpha)$ as the subclass of $\mathcal{A}$ consisting of all starlike functions of order α in $\mathcal{D}$.

Note that a function $f\in\mathcal{A}$ is said to be starlike in $\mathcal{D}$ when $\alpha=0$.

Definition 1.5 (Goodman, 1975) A function $f(z)$ in the form (1.1) is said to be convex of order α in $\mathcal{D}$ if for all $z\in\mathcal{D}$

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \quad z\in\mathcal{D}$$

where $0\leq\alpha<1$. We denote $\mathcal{C}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all convex functions of order α in $\mathcal{D}$.

Note that a function $f\in\mathcal{A}$ is said to be convex in $\mathcal{D}$ when $\alpha=0$.

The concepts of starlike and convex functions of order α were introduced by Robertson (1936). Thus, many other mathematicians continue to study and investigate this idea.

Definition 1.6 (Goodman, 1975) A function $f(z)$ in the form (1.1) is said to be strongly starlike of order α in $\mathcal{D}$ if for all $z\in\mathcal{D}$,

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2}$$

for $0<\alpha\leq 1$.

The set of all such functions is denoted by $\mathcal{SS}^*(\alpha)$.

Definition 1.7 (Goodman, 1975) A function $f(z)$ in the form (1.1) is said to be strongly convex of order α in $\mathcal{D}$ if for all $z\in\mathcal{D}$,

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\alpha\pi}{2}$$

for $0<\alpha\leq 1$.

The set of all such functions is denoted by $\mathcal{SC}^*(\alpha)$.

Definition 1.8 (Noor & Thomas, 1980) Let f be analytic in $\mathcal{D}$ with $f(0) = f'(0) - 1 = 0$. Then f is said to be quasi-convex in $\mathcal{D}$ if there exist a function $g(z) \in \mathcal{C}$ with $g(0) = g'(0) - 1 = 0$ such that for $z \in \mathcal{D}$

$$Re\left(\frac{(zf'(z))'}{g'(z)}\right) > 0$$

The set of all such functions is denoted by $\mathcal{Q}(\alpha)$.

1.4 Bi-Univalent Functions

According to the Koebe one-quarter theorem by Duren (1983), the image of $\mathcal{D}$ under every univalent function $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus, an inverse for every univalent function f can be defined as

$$f^{-1}\big(f(z)\big) = z, \quad z \in \mathcal{D}$$

and

$$f\big(f^{-1}(w)\big) = w, \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right)$$

where

$$\begin{aligned} f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 \\ - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \end{aligned} \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in D if both f and f^{-1} are univalent in $\mathcal{D}$. The class of bi-univalent functions defined in $\mathcal{D}$ is denoted by σ. Some example of functions in class σ include $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.

The first researcher to introduce and study about the class of bi-univalent functions is Lewin (1967). He proved that $|a_2| < 1.51$. Later on, Brannan and Clunie (1980) refined the result to $|a_2| < \sqrt{2}$. While Netanyahu (1969) showed that $\max_{f\in\sigma}|a_2| = \frac{4}{3}$.

Brannan and Taha (1986) instigated certain subclasses of bi-univalent functions. It has similarities with the subclasses of univalent functions which consisting of starlike, convex and strongly starlike functions. They investigated bi-starlike functions and obtained the estimates on the initial coefficients.

There are two main subclasses of bi-univalent functions which are bi-starlike and bi-convex functions. These subclasses are denoted as $\mathcal{S}_\sigma^*$ and $\mathcal{C}_\sigma$, respectively.

1.5 Subordination

Definition 1.8 (Goodman, 1975) Let $G(z) = a_0 + a_1 z + \cdots$ be analytic and univalent in $\mathcal{D}$ and suppose that $F(\mathcal{D}) = \delta$. If $f(z)$ is analytic in $\mathcal{D}$, $f(0) = G(0)$, and $f(\mathcal{D}) \subset \delta$, then we say that $f(z)$ is subordinate to $G(z)$ in $\mathcal{D}$, and we write

$$f(z) \prec G(z)$$

We also say that $G(z)$ is superordinate to $f(z)$ in $\mathcal{D}$.

1.6 Functions with Positive Real Part

The convex and starlike functions are relatively related to functions with positive real part. We give definition to these functions as follow.

Definition 1.9 (Goodman, 1975) The set $\mathcal{P}$ is the set of all functions of the form

$$\mathcal{P}(z) = 1 + p_1 z + p_2 z^2 + p_n z^n \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n$$

that are analytic in $\mathcal{D}$, and such that for z in $\mathcal{D}$, $Re(\mathcal{P}(z)) > 0$. Any function in $\mathcal{P}$ is called a function with positive real part in $\mathcal{D}$.

In this case, it should be noted that $\mathcal{P}(z)$ is not required to be univalent. Thus, $\mathcal{P}(z) = 1 + z^n$ is in $\mathcal{P}$ for any integer $n \geq 0$, but if $n \geq 2$, this function is not univalent.

Just as the Koebe functions, $k(z)$ plays a major role in the class $\mathcal{S}$, the Möbius function

$$L_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots = 1 + 2\sum_{n=1}^{\infty} z^n, \qquad z \neq 1$$

Also plays a major role in the class $\mathcal{P}$. This function is belonging to the class $\mathcal{P}$, it is analytic and univalent in $\mathcal{D}$, and its maps $\mathcal{D}$ onto the half plane.

1.7 Objectives of Research

The objectives of this research are:

i. to obtain the estimates of the Maclaurin coefficients, $|a_2|$ and $|a_3|$ for functions belonging to certain subclasses of σ;

ii. to obtain the upper bounds of the Fekete-Szegö functional for functions belonging to certain subclasses of σ; and

iii. to obtain the upper bounds of the second Hankel determinant for functions belonging to a subclass of σ.

1.8 Outline of Research

This thesis consists of five main chapters. Chapter 1 explains and gives brief insight of the ideas of the geometric function theory, analytic and univalent functions and its subclasses, bi-univalent functions and its subclasses and functions with positive real part. Some definitions are used to substantiate these terms. Chapter 2 consists of defining certain subclasses of bi-univalent functions which are denoted by $\mathcal{H}_\sigma(\varphi,\alpha)$, $\mathcal{I}_\sigma(\varphi)$, $\mathcal{L}_\sigma(\varphi)$ and $\mathcal{K}_\sigma(\varphi,\alpha)$. Chapter 3 mentions about the Fekete-Szegö functional for all the certain new subclasses of σ. Chapter 4 mentions about the results on the upper bounds of the second Hankel determinant for a new subclass of σ. Lastly, Chapter 5 gives the conclusion of the thesis and future works of this study.

CHAPTER 2

INITIAL COEFFICIENTS

2.1 Introduction

In Chapter 1, brief history of functions in class σ have been discussed while in this chapter, several new subclasses of σ will be developed. These subclasses will then be denoted by $\mathcal{H}_\sigma(\varphi,\alpha)$, $\mathcal{I}_\sigma(\varphi)$, $\mathcal{L}_\sigma(\varphi)$ and $\mathcal{K}_\sigma(\varphi,\alpha)$ with $0 \le \alpha \le 1$. The ideas of developing new subclasses of σ were inspired by some well-known authors such as Ali *et al.* (2012), Srivastava *et al.* (2013) and Lashin (2016).

2.2 The Class $\mathcal{H}_\sigma(\varphi,\alpha)$

Before giving the definition of the new subclass of σ, we begin by stating the well-known definition of a class which was introduced by Ma and Minda (1994).

Definition 2.1 (Ma & Minda, 1994) considered an analytic function φ with positive real part in $\mathcal{D}$, $\varphi(0) = 1$, $\varphi'(0) > 0$ that maps $\mathcal{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The series expansion for function φ can be expressed in the form of

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \quad (B_1 > 0) \tag{2.1}$$

The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \tag{2.2}$$

and the class of Ma-Minda convex functions consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \tag{2.3}$$

respectively. Ali *et al.* (2012) stated that in order for function f to be bi-starlike or bi-convex of Ma-Minda type, both f and f^{-1} must be respectively Ma-Minda starlike or Ma-Minda convex. $\mathcal{ST}_\sigma(\varphi)$ and $\mathcal{CV}_\sigma(\varphi)$ denote the classes for bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type, respectively.

Motivated by the classes $\mathcal{ST}_\sigma(\varphi)$ and $\mathcal{CV}_\sigma(\varphi)$, we come out with the subclass of σ which is denoted by $\mathcal{H}_\sigma(\varphi, \alpha)$ as follows.

Definition 2.2 A function $f \in \sigma$ given by (1.1) is said to be in the class $\mathcal{H}_\sigma(\varphi, \alpha)$ with $0 \leq \alpha \leq 1$ if the following subordination hold:

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec \varphi(z) \tag{2.4}$$

and

$$\frac{wg'(w) + \alpha w^2 g''(w)}{(1-\alpha)g(w) + \alpha w g'(w)} \prec \varphi(w) \tag{2.5}$$

where the function g is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^2 - \cdots \tag{2.6}$$

In particular, for $\alpha = 0$, the class of $\mathcal{H}_\sigma(\varphi, \alpha)$ is reduced to the class $\mathcal{ST}_\sigma(\varphi)$, bi-starlike of Ma-Minda type and for $\alpha = 1$, the class of $\mathcal{H}_\sigma(\varphi, \alpha)$ is reduced to $\mathcal{CV}_\sigma(\varphi)$, bi-convex of Ma-Minda type.

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