MAKING SENSE OF COMPLEX NUMBERS

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DECLARATION

I hereby declare that the material in this thesis is my own except for quotations, excerpts, equations, summaries and references, which have been duly acknowledged.

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CERTIFICATION

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ABSTRACT

This research exemplifies a framework of mathematical thinking as proposed by Chin & Tall (2012) and Chin (2013) by showing the empirical evidence related to how a group of participants made sense of complex numbers. Humans make sense of new mathematics context by relating it to personal conceptions and experiences. This might not be a smooth process as it may involve conceptions that work in an old context but do not work in a new context. The researcher has chosen the topic of complex numbers as it is an extension of the real numbers system and it will be beneficial to explore how the conceptions of participants in the real numbers system affect their sense making of complex numbers. This study involves thirty undergraduate students majoring in mathematics from the faculty of Science and Natural Resources, Universiti Malaysia Sabah. After analysing the responses of the participants for the given mathematical task, a follow-up interview was conducted with five participants. The interview session was recorded and transcript were made of the interviews. The transcripts were then analysed. The results show that these participants have supportive and problematic conceptions in making sense of complex numbers and these conceptions are originated from the context of real numbers.

ABSTRAK

MEMAHAMI NOMBOR KOMPLEKS

Kajian ini bertujuan untuk mengesahkan kerangka pemikiran matematik yang dicadangkan oleh Chin & Tall (2012) dan Chin (2013) dengan menunjukkan bukti empirikal tentang bagaimana sekumpulan responden membina pengertian (make sense) tentang nombor kompleks. Manusia membina pengertian tentang suatu konteks matematik yang baharu dengan menghubungkaitkan konteks tersebut dengan pengetahuan konsep dan pengalaman peribadi mereka. Jika suatu konsep boleh digunakan dalam konteks yang lama tetapi tidak boleh diguna-pakai dalam konteks yang baharu, perkara ini akan menyebabkan proses pembelajaran terhalang. Pengkaji telah memilih topik nombor kompleks kerana ia merupakan lanjutan daripada sistem nombor nyata dan pengkaji berpendapat bahawa kajian ini akan menghasilkan suatu ilmu pengetahuan yang bermanfaat apabila penerokaan tentang bagaimana pengertian konsep responden di dalam sistem nombor nyata mempengaruhi pengertian konsep mereka di dalam sistem nombor kompleks. Kajian ini melibatkan tiga puluh orang pelajar yang mengikuti kursus Matematik daripada fakulti Sains dan Sumber Alam, Universiti Malaysia Sabah. Selepas analisis dilakukan bagi kertas kaji selidik yang telah dilengkapkan oleh responden, satu sesi temu bual dijalankan bersama lima responden terpilih. Setiap sesi temu bual tersebut telah direkodkan dan satu transkrip telah dibuat. Transkrip itu kemudiannya dianalisis. Dapatan kajian menunjukkan bahawa responden-responden ini mempunyai pelbagai pengertian konsep yang berasal daripada konteks nombor nyata yang menyokong (supportive) atau menjadi halangan (problematic) dalam proses membina pengertian konsep dalam konteks nombor kompleks.

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CHAPTER 1

INTRODUCTION

1.1 Study Background

The imaginary number, i took centuries of time to convince mathematician to accept it as a part of the number system. Despite the word imaginary in its name, the imaginary numbers are not imaginary at all. Imaginary numbers were invented in an effort to evaluate negative square roots, a mathematical operation, that before the existence of imaginary number was deemed impossible to solve. Thus, the argument to justify the existence of imaginary numbers is similar to the argument for the existence of integers, rational numbers, and real numbers. Leonhard Euler, an eighteenth-century Swiss mathematician, known for his prolific writing in mathematics and his standardization of modern mathematics notation, chose the symbol i to stand for the square root of -1 (Merzbach & Boyer, 2011, p.408). Since then, the imaginary number set has included i and any real numbers times i.

It is interesting and important to observe the behavior of i when it is multiplied by itself. To begin, i multiplied by itself is i^2. By the definition of imaginary number, we know that $i^2 = -1$. To continue this line of reasoning, $i^3 = -i$. Similarly, $i^4 = 1$ (Hom, 2014). Continuing, we will find that $i^5 = i$, and this is where the pattern begins again ($i, -1, -i, 1$). Incredibly, by using imaginary numbers, it is possible to solve many equations that were deemed impossible to solve for centuries.

After the imaginary number was created, subsequently the complex numbers which contain both real part and imaginary part was developed. The complex numbers are introduced primarily in order to extend the notion of roots of all quadratic equations (Chaves, 2014). Thus, the real numbers and imaginary number are subset of the complex numbers. Complex numbers are numbers in the form of $a+bi$, where *a* is the real part and b is the imaginary part. The real numbers, ℝ are the complex numbers in the form of a+0i, in which the imaginary part is equal to zero whereby the imaginary number, i is a complex number in the form of 0+bi where the real part as opposed to the real numbers is equal to zero (Serdarushich, 2019).

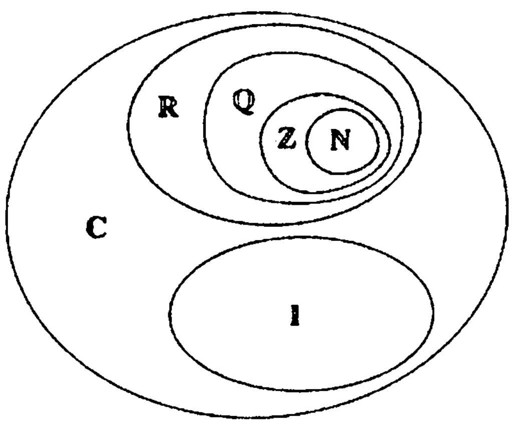

DIAGRAM 1: Set Representation of Number Systems

The rule and conception of number system is usually applicable when used in a broader context, for example (refer to Diagram 1), Natural numbers, N is a subset of rational numbers, Q. The conception in natural numbers still works in the rational numbers (e.g. inequality). This, however, is not the case when we move from real numbers to the complex numbers. There are some rules that do not apply in complex numbers and this will conflict with the learners' existing knowledge. This study aims to discuss on this issue. The work by Chin and Tall (2012) which proposed the idea of supportive and problematic conception in making sense of trigonometry has inspired and gives a clear framework for the researcher to study complex numbers and determine the supportive and problematic conception involved in the process of making sense of it.

The learning content in Malaysian Curriculum is arranged from basic to a more advanced knowledge. Thus, learning Mathematics requires individuals to learn the basic concept first before they advance to a more sophisticated Mathematics topic. This is because the learning process occurs by connecting links between any related experiences in our mind, and the learners build experiences based on the things they have met-before (Lima & Tall, 2008). *Met-before* is defined as a mental construct that an individual uses at a given time based on what they have met-before (Tall 2004b). In this research, The researcher focus on the preconception of complex numbers and aims to show the close relationship that exists between met-before and preconception. Preconception is an important element in learning Mathematics because learners build their understanding by trying to make sense of what they have learnt. Preconception can also be seen as the formal or informal form of conception that individuals have in their minds. Vinner and Tall (1981) state that, we gather formal form of preconception based on what we have learnt in the classroom and it usually has a formal name, while the informal form of preconception may just be a mental image based on daily life experience.

The researcher will classify and divide preconception into two: supportive conception and problematic conception as suggested by Chin and Tall (2012). Supportive conception helps the learners when they are in the process of learning new things, while the problematic conception will interfere and impede the learning process. Further in this chapter, the researcher will discuss about the problem statement, research objectives, research questions, significance of the study, limitation of the study, operational definition and the summary of this chapter.

1.2 Problem Statement

The teaching and learning process of Mathematics has always been a challenge, both to the instructor and learners. This is due to the fact that both individuals have gone through a different learning experience. Vinner and Tall (1981) introduce the terms concept image to describe the total cognitive structure that is associated with the concept. They argue that a complex cognitive structure exists in the mind of every individual, thus, yielding a variety of personal mental images when a concept is evoked. In the learning of Mathematics, the mental images that exist in every individual can also be classified as a preconception which supposedly aid or support the learning process, thus, making it easier for the learner to understand a new concept. This might not be the case for all individuals. They may also have mental images that prevent them from understanding a concept, because it conflicted with another mental images (Vinner & Tall, 1981). This is what Chin and Tall (2012) classified as the problematic conception because it impedes the learning process.

Focusing on the supportive conception and problematic conception allows us to look at the misconception from a different angle. The term misconception is often used to point out a wrong conception that is applied in solving a mathematical problem. There is a wide usage of the term misconception in several studies by other researchers, and to name one of it is a study related to Mathematics by Bakar and Tall (1992c). They used the term misconception to describe the mistakes of their respondents. Danenhower (2000) also mentioned misconception in his study about teaching and learning complex numbers in two British Columbia universities. Moreover, Nordlander and Nordlander (2012) also applied the term misconception in their study about the concept image of complex numbers.

Previous researchers labeled conceptions that are wrongly applied in a certain context as misconception. By doing this and continue using the term misconception without conducting further investigation about the cause that triggers the "said" misconception, we are actually limiting our view of the bigger issues. The concept itself may not be wrong at all but I believe that there is a high possibility that the concept is just been used in the wrong context which in the end producing an incorrect response for certain mathematical problems. Therefore, by conducting this study, I hope to identify the source of the supportive conception, the problematic conception and how the problematic conceptions affect the sense making of complex numbers. I believe that if the source of the supportive conception and problematic conception that exist as a mental image of a learner can be traced, then, it will definitely aid the process of making a better teaching strategy for complex numbers and in the long run will promote better understanding of complex numbers among learners.

1.3 Research Objectives

The objective of this study is to explore:

1. The supportive conception in the sense making of Complex numbers.
2. The problematic conception in the sense making of Complex numbers.
3. The effect of problematic conception in the process of sense making of Complex numbers.
4. The evoked concept image of complex numbers.

1.4 Research Questions

This study is conducted to answer the questions below:

1. What are the supportive conceptions in the sense making of Complex numbers?
2. What are the problematic conceptions in the sense making of Complex numbers?
3. How problematic conceptions affect the sense making process of Complex numbers?
4. What is the evoked concept image of complex numbers?

1.5 Significance of the Study

The Malaysian curriculum developers will benefit from this study as the current system only focuses on supportive conception. It overlooks certain issues that occur when a concept is applied across the context. For example, the conception of triangle in Euclidean geometry impedes the generalization of triangle in the Cartesian plane because the sides and angles, which previously had specific properties as unsigned quantities, can now be in any size, positive or negative (Chin & Tall 2012). Such knowledge can be used to construct a method to counter the effect of the problematic conception in the process of learning. The analysis of students' thinking is seen as a resource that can help teachers make informed decisions in their classrooms and improve their practice. Such a listening orientation towards teaching promotes a learning environment conducive to and respectful of students' own sense making and intellectual autonomy (Davis, 1996; Kamii, 1989).

The researcher hopes that this study will be a platform to inform the curriculum developers that a curriculum can be built based on problematic conception and method to counter problematic conception can be developed in the learning of Mathematics. Makgakga (2014) states that it is imperative for teachers to teach Mathematics using learners' errors and misconceptions as this will guide them on what learners grapple with. One of the ways to implement this idea in our education system is to create or devise a new course that emphasizes on the problematic conception for the prospective teachers so that they will be able to guide the learners.

1.6 Limitations of Study

The limitations of this study are:

1. The supportive and problematic conceptions are limited only to the topic of Complex numbers.
2. This study is classified as an empirical study which conducted in Universiti Malaysia Sabah. The informants are thirty undergraduates with complex numbers background who enroll in Mathematics with Economics Course.
3. The outcome of this study cannot be used to generalize other population and this study is bounded to the limitation stated above.

1.7 Operational Definition

The operational definitions used in this study are:

1. **Met Before**

   A current structure resulting from earlier experiences. It refers not to the actual experience itself, but to the trace that it leaves in the mind that affects our current thinking. (Tall 2004a).

2. **Mental Pictures**

   Mental picture is defined as the met before that relates to the thing that we discussed (Vinner & Tall 1981).

3. **Concept Image**

   Total cognitive structure that is associated with the concept which includes all the mental pictures and associated properties and processes (Vinner & Tall 1981).

4. **Supportive Conception**

   The conception that support learning process and generalization in a new context (Chin & Tall 2012).

5. **Problematic Conception**

   The conception that impede learning process and generalization. (Chin & Tall 2012).

6. **Evoked Concept Image**

   Portion of the concept images that are activated at a particular time (Vinner & Tall 1981).

7. **Complex Numbers**

   Complex numbers is a numbers in the form of $a + bi$, where $a =$ real part and $b =$ imaginary part. $a, b \epsilon \mathbb{R}$

1.8 Summary

This chapter discussed the introduction for this research such as the study background, the problem statement, the research objectives, the research questions, the significance of the study, the limitation of the study, the operational definition and the summary of chapter 1. According to the literature reviews, the preconception from the real numbers context will affect the process of sense making of complex numbers especially the concept of ordering of numbers in the context of real numbers. Tirosh and Almog (1989), in their study, it shows that 95% of the students agreed that the following inequality $i < 4 + i$ is true, where i is an imaginary number. The students have explained that when a positive number is added to a number, it makes the number larger, so they conclude $i < 4 + i$. Danenhower (2000), in his study also portrays resemblance to the study by Tirosh and Almog, that the students have problem understanding that the ordering relation in real numbers does not extend to complex numbers. This is one of the area that the researcher hopes to find a similar result as the previous researchers did. Finding literatures regarding the complex numbers prove to be a challenging task, and not to add that the same can be said about finding past studies that focus on the progression of conception from real numbers to complex numbers. This situation motivates the researcher to do a research on the topic of complex numbers focusing on the conceptions involved in both real and complex numbers contexts. The researcher is interested to explore conceptions that are extendable from real numbers context to the complex numbers context as well as the conceptions that are not. To execute this research, the researcher uses the framework of mathematical thinking by Chin and Tall (2012) which has a different viewing angle and focus more on the source of the preconception of the informant and classify it as either supportive or problematic conceptions. The researcher refuses to dismiss such mistake by the informants in their process of solving and reasoning a mathematical related problem using the wrong concept as a mere "misconception". I believe that there are more stories to be told behind this so called "misconception", thus leading me to choose this framework for my research.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This study is conducted to determine the supportive conception, problematic conception and how problematic conception affects the sense making of complex number by the learners. Learners are exposed to the knowledge of number systems throughout their study period. In Malaysia, learners learn the real number prior to complex number from primary school until they complete their secondary schooling. After completing the five-year secondary schooling, they will learn about complex number in their pre-university level. Imagine trying to find a solution of $x + 6 = 4$ but only being able to look for a solution in the set of Natural numbers (number system with only positive whole number), which is impossible. However, if we expand our domain to integer (number system containing both positive and negative whole number), -2 provides a solution. Similarly, it is impossible to find a solution to $2x = 7$ when working only in the domain of integer, but we can expand our domain to the set of rational number (number system that contains integer and number in the form of fractions), and thus we have 3.5 or $\frac{7}{2}$ as a solution. Now suppose you want to find a solution to $x^2 = 2$ using only rational numbers. This, too, is impossible. However, the set of rational number can be expanded to create still another new set of numbers—the real numbers. Clearly, there is one solution to the equation $x^2 = 2$ because by definition, the square root of any number multiplied by itself equals to the number. Another solution is when the negative square root of a number multiplied by itself equals the number ($\sqrt{-x} \times \sqrt{-x} = -x$).

In tertiary education, for instance in college and university, the real number system will be extended to complex number system. This is where the researcher expects to find the pre-conception that either support or impede the learning of complex number.

The number system itself has different context like the real number, R and the complex number, C. Following the work of Chin and Tall (2012), the researcher aims to investigate the pre-conception involved in the learning of complex numbers. This chapter will present the history of complex number, the chronological development of conceptions, past research studies, conceptual framework and the summary of the chapter.

2.2 History of Complex Number

The first introduction of imaginary numbers occurs in connection with the solution of cubic equations. During the 16th century, Italian mathematicians of the famous University of Bologna discovered that sometimes correct answers could be obtained more expeditiously if they assumed: a symbol i, $i^2 = -1$ and in other respect treated i as an ordinary number (Diamond, 1957).

Euler (1707 - 1783) achieved brilliant results by the use of complex numbers but the fundamental principle of their logic was either not deemed important or was completely misunderstood.

Wessel as cited in Diamond (1957) a Danish mathematicians probably furnished the first logical foundation for complex numbers but his work received very little of attention.

Gauss (1777 - 1855) was the first to use the phrase complex number, the symbol i, for imaginary unit and introduced mathematicians to the true theory of these numbers. Gauss' work was followed by Cauchy and Riemann (1826 - 1866).

2.3 Conceptions

The study of conception has been conducted decades ago and it is still developing and generating a new way on how to look at the term "conception". Vinner and Tall (1981) introduce the terms concept image and concept definitions. Concept image represents the total cognitive structure that relates to the concept, while, concept definition is used to designate the formal mathematical definition that explains the concept. We can surmise that, concept definition should be identical for everybody and does not change from one individual to another, whereas the concept image differs between individuals and is reflecting personal reconstructions of a definition which generates individual perceptions. Sfard (1991) proposes a similar idea by referring it as conception. The whole cluster of internal representations and associations evoked by the concept – the concept's counterpart in the internal, subjective "universe of human knowing". The personal mathematical conception has a subjective nature which cause the difference in the performance of every learner.

Tall (2004) introduces the term met-before to represent the current structure resulting from a past experience. Later, Lima and Tall (2008) suggest that the previous experience which is known as met-before will have an effect on new context. Met-before highlights the importance of previous experience in shaping mathematical conception. There might be supportive met-before and problematic met-before incorporated in conceptions. A supportive met before is defined as a kind of experience which supports the development of coherent knowledge structure in a new situation. In contrast, a problematic met-before impedes generalization in new situation. It should be noted that supportive or problematic met-before can be incorporated into new conceptions, for instance, the met-before of 'take-away always gives less' in natural number is regarded as a supportive met-before when working in positive integers. On the other hand, this met-before will become problematic when working in the context of negative integers.

Chin and Tall (2012) later suggest that these met-before can be considered as supportive conception and problematic conception. Supportive conception will help to generalize existing knowledge to a new context while problematic conception can impede progress because they become an obstacle or conflict when the existing knowledge does not work in the new context. Both supportive conception and problematic conception are closely linked on how a person is making sense of mathematics. It also opens up new perspective in viewing misconceptions. They also added, rather than recognizing these difficulties as

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