Implementation of the 4EGKSOR for Solving One-Dimensional Time-Fractional Parabolic Equations with Grünwald Implicit Difference Scheme

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Abstract. Solving one-dimensional time-fractional parabolic equations using numerical technique will require some iterative solver to solve the generated large and sparse linear systems. Thus, by considering the advantages of the Explicit Group iteration technique together with the Kaudd SOR (KSOR) iterative method, this paper examines the efficiency of the four-point Explicit Group Kaudd SOR (4EGKSOR) iterative method to solve the approximation equations generated from discretization of one-dimensional time-fractional parabolic equations using the finite difference scheme with the second order Grünwald Implicit difference scheme. In addition, the formulation and implementation of the proposed method to solve the problem are also presented. Numerical result and comparison with four-point Explicit Group Gauss-Seidel (4EGGS) method are given to illustrate the efficiency of the proposed method.

Keywords: Time-Fractional Parabolic equation, Grünwald derivative, Implicit Finite Difference, 4EGKSOR iteration.

1 Introduction

Fractional differential equations (FDEs) have profound applications in the fields of sciences, engineering and economics [1-3]. Some of the benefits of fractional derivatives have been mentioned in many researches. It is said to be better in describing some type of real-life phenomena [4] since users have freedom to select the order of the derivative [5] as compared to the ordinary derivative. This also made it useful in problem solving related to non-Markovian random walk [6], which involves systems with long-term memory.

In this study, we considered inhomogeneous Time-fractional parabolic equations (TFPE) which can be defined in general form as follows

$$\frac{\partial^{\alpha} U}{\partial t^{\alpha}} + p(x)\frac{\partial U}{\partial x} = q(x)\frac{\partial^2 U}{\partial x^2} + f(x,t), \qquad a \le x \le b, \quad 0 \le t \le T, \quad 0 < \alpha \le 1, \tag{1}$$

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where the function *f(x,t)* is a source term and note that for α=1, equation (1) is the classical parabolic partial differential equation.

There are plenty of recent development in numerical approach for solving problem (1) could be found in literatures, such as refer [7]-[10]. Numerical treatment of TFPE will generate approximation equations with a character of large and sparse system of linear equations (SLE). So, in order to reduce the computational complexity, our interest is to solve the generated SLEs using iterative method. Previously, similar studies to us could be found such as [11]-[12]. However, in most of their studies, the fractional derivatives are defined in the sense of Caputo, while in this study we discretized problem (1) using Grünwald fractional operator and implicit difference scheme. The definition of Grünwald fractional derivative we used in this study is defined by [13]-[14].

Definition 1 The Grünwald fractional derivative of order α of a function *f(t)* is defined as

$$D_G^{\alpha} f(t) = \frac{1}{(\Delta t)^{\alpha}} \lim_{N\to\infty} \sum_{k=0}^{j} g_{\alpha,k} f(t - k\Delta t), \qquad 0 < \alpha < 1 \tag{2}$$

where the Grünwald weights are $g_{\alpha,k} = \dfrac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}$.

One of the well-known classical point-iterative method is the Gauss-Seidel (GS) method. To accelerate the convergence rate even further, Evans [15] introduced the Explicit Group (EG) method, which have been investigated extensively over the past years. Recently, inspired by the same objective, Youssef [16] have introduced a new method which is called as Kaudd Successive Over-Relaxation (KSOR) iterative method. This method is a modification of the classic SOR iterative method, which the difference lies on the values of the relaxation parameter allowed. Recently, the effectiveness of the KSOR method has also been discussed to solve the integral equation [17] and two-point boundary value problem in [18].

Thus, in this paper our focus is to investigate the effectiveness of the four-point Explicit Group (4EG) method together with the KSOR iterative method, in solving problem (1) since none of the previous researches have employed this technique to solve such fractional equation problem before. Throughout this paper, we will refer this iterative method as 4EGKSOR method. For simplicity purposes, the solution domain (1) is assumed to be uniformly divided into $N = 2^p$, p ≥2 in directions of x and t, which the subintervals are denoted as Δx and Δt respectively and defined as

$$\Delta x = \frac{(b-a)}{N} = h, \quad n = N-1, \qquad \Delta t = \frac{T}{M} \tag{3}$$

This paper is outlined as such. Next section we discuss the derivation of second-order Grunwald implicit approximation equations for problem (1) followed by the formulation of 4EGKSOR iterative method in Section 3. In Section 4, numerical re-