

**USING THE LINE ARRAY REPRESENTATION
AS AN ALTERNATIVE METHOD FOR
UNDERSTANDING WHOLE NUMBER
MULTIPLICATION AMONG
PRIMARY STUDENTS**

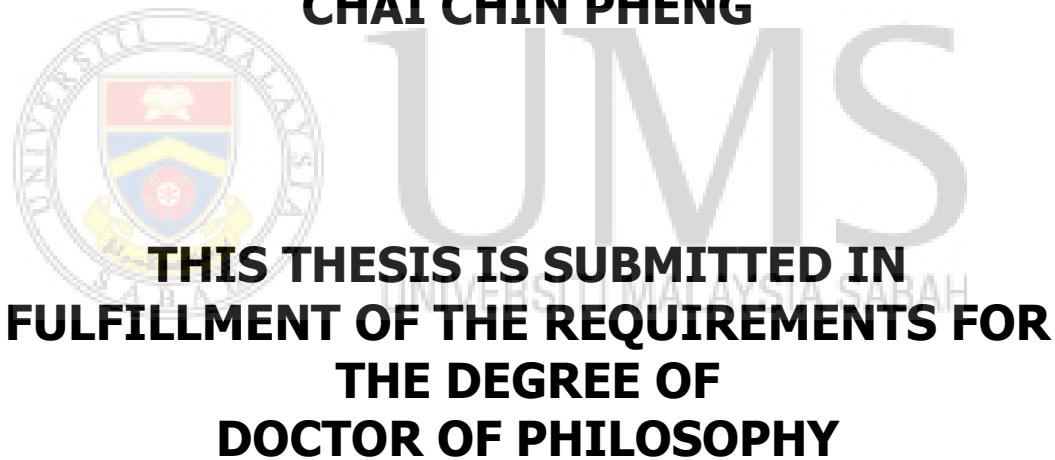


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CHAI CHIN PHENG
UNIVERSITI MALAYSIA SABAH

**FACULTY OF PSYCHOLOGY AND EDUCATION
UNIVERSITI MALAYSIA SABAH
2015**

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2015**

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DECLARATION

I hereby declare that the material in this thesis is my own except for quotations, excerpts, equations, summaries and references, which have been duly acknowledged.

20 August 2015

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CERTIFICATION

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DEGREE : **DOCTOR OF PHILOSOPHY (MATHEMATICS EDUCATION)**

VIVA DATE : **20 AUGUST 2015**



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ABSTRACT

This study aims to examine how Line Array Representation can support primary school students' understanding in single digit multiplication (SDM) and multi-digit multiplication (MDM) whole number calculations among Year 3 and Year 5 students as well as to determine its effect on the students' understanding of whole number multiplication through a multiplication test for each level constructed based on the range of calculation strategies associated with multiplication. A theoretical framework in mathematical growth which connects between the embodiment and symbolism domains was developed to examine the understanding in multiplication. The construction of multiplication concept can be examined through whether student is able to perceive, act and reason upon the line array that is considered to be the key representation for multiplication in this study. Having established the theoretical framework, the research design consists of a mixed methods approach was drawn up to conduct a qualitative study in the first phase and followed by a quantitative study in the second phase. The design is based on 'QUAL → quan Sequential Exploratory Design' which has given priority to the qualitative aspect of the study. The participants of this study were taken from an intact class of 29 Year 3 (15 boys, 14 girls) and 32 Year 5 (19 boys, 13 girls). Year 5 data was collected during an intervention in early February to mid-March 2014 while Year 3 data in mid-March to end of April 2014. Qualitative data was collected by using the teaching experiment technique. Students worked in pairs on ten SDM and two MDM tasks. Analysis of the obtained audio- and video-recordings, written tasks and field notes had identified strategies in which the Line Array Representation supported or hindered students' multiplication calculations. Results showed that students were able to apply the commutative and distributive over addition property to solve multiplication problem, most evident in 2-digit by 2-digit multiplication. Using line array, students were capable of inventing own procedures, solving MDM from the left to the right (thousands, hundreds, tens and units) obviously opposes the traditional algorithm of working from right to left. The quantitative data was collected from the 33-item multiplication test which was administered to each student before and after the intervention. The data obtained from the tests were subjected to Rasch analysis using Quest software. Overall, the resulting measures have shown to have good reliability as indicated by a Cronbach's alpha value of 0.79 for Year 3 and 0.82 for Year 5. The higher mean person estimates (Year 3 = +0.27 logits and Year 5 = +0.98 logits) compared with the mean item threshold defaulted at 0.00 logit showed that items were likely well matched for Year 3 students and easy for Year 5 students. Problems that students had faced with using the array and dilemmas that researcher encountered throughout the study are discussed. Based on these findings, two implications of this study are (1) pedagogical use of visual representations by teachers and (2) the benefits of using Item-Response Theory (IRT) model by teachers in assessments. Recommendations for further improvements in the test, structure of the line array and a different method of data collection are also put forward.

ABSTRAK

MENGGUNAKAN PERWAKILAN SUSUNAN GARISAN SEBAGAI KAEADAH ALTERNATIF DALAM KEFAHAMAN PENDARABAN NOMBOR BULAT DALAM KALANGAN MURID SEKOLAH RENDAH

Kajian ini bertujuan untuk meneliti bagaimana perwakilan susunan garisan boleh menyokong murid dalam pendaraban nombor bulat satu digit dengan satu digit (SDM) dan pendaraban berbilang digit (MDM) murid Tahun 3 dan Tahun 5 serta untuk menentukan kesannya terhadap kefahaman murid tersebut tentang pendaraban nombor bulat melalui ujian Pendaraban yang dibina bagi setiap tahap berdasarkan pelbagai strategi pengiraan yang berkaitan dengan pendaraban. Dengan mengadaptasikan kerangka teori yang melibatkan perkembangan matematik, kefahaman konsep pendaraban bagi murid sekolah rendah dapat dilihat sebagai persilangan di antara domain penjelmaan dan simbolisme dan di mana strategi pengiraan pendaraban akan muncul di dalam persilangan kedua-dua domain tersebut. Pembinaan konsep pendaraban boleh dikaji melalui sama ada murid dapat melihat, bertindak dan menaakul; perwakilan susunan garisan yang dianggap sebagai perwakilan utama bagi pendaraban dalam kajian ini. Setelah mengukuhkan kerangka teori, rekabentuk penyelidikan yang terdiri daripada pendekatan kaedah gabungan digunakan untuk menjalankan satu kajian kualitatif dalam fasa pertama dan diikuti oleh kajian kuantitatif dalam fasa kedua. Rekabentuk kajian adalah berdasarkan rekabentuk eksploratori bersiri 'QUAL►quan' di mana keutamaan analisis data diberi kepada aspek kualitatif dalam kajian ini. Peserta kajian ini terdiri daripada satu kelas Tahun 3 seramai 29 orang (15 lelaki, 14 perempuan) dan satu kelas Tahun 5 seramai 32 orang murid (19 lelaki, 13 perempuan), di mana tahap kebolehan murid telah dibahagikan kepada tiga tahap yang berbeza; tinggi, sederhana dan rendah. Data Tahun 5 dikumpul dari awal Februari hingga pertengahan Mac 2014 manakala data Tahun 3 dari pertengahan Mac hingga akhir April 2014 melalui intervensi seperti yang dirancangkan. Data kualitatif dikumpul dengan menggunakan teknik Eksperimen Pengajaran, di mana murid secara berpasangan menyelesaikan tugas yang terdiri sepuluh sepuluh SDM dan dua MDM tugas dalam lembaran kerja yang disediakan. Analisis daripada data kualitatif yang diperolehi secara audiovisual dan bertulis telah mengenalpasti beberapa strategi yang digunakan oleh murid menunjukkan perwakilan susunan garisan dapat menyokong murid dalam pengiraan pendaraban. Dapatan kajian menunjukkan bahawa murid mampu mengaplikasikan Hukum Tukar Tertib dan Taburan dalam penambahan untuk menyelesaikan masalah pendaraban, paling ketara ditunjukkan dalam pendaraban 2-digit dengan 2-digit. Menggunakan perwakilan garisan tersebut, murid berupaya mencipta prosedur terutama dalam menyelesaikan MDM, iaitu mengira dari kiri ke kanan (ribu, ratus, puluh dan sa). Cara pengiraan ini jelas menunjukkan ia bertentang dengan algoritma yang tradisional, iaitu bekerja dari kanan ke kiri. Data kuantitatif pula dikutip daripada skor ujian pendaraban yang terdiri daripada 33 item telah ditadbirkan kepada setiap murid sebelum dan selepas intervensi. Data yang diperolehi daripada ujian-ujian tersebut telah dianalisis dengan menggunakan analisis Rasch, perisian Quest. Secara keseluruhan, dapatan kajian menunjukkan

kebolehpercayaan item yang baik, di mana nilai alfa Cronbach adalah 0.79 untuk Tahun 3 dan 0.82 untuk Tahun 5. Min anggaran kebolehan murid yang lebih tinggi (Tahun 3 = +0.27logits dan Tahun 5 = +0.98 logits) berbanding dengan min item (0.00 logit) menunjukkan kemungkinan bahawa item-item adalah berpadanan dengan tahap kebolehan murid Tahun 3 manakala agak mudah untuk murid Tahun 5. Tambahan pula, semua peta 'Item-fit' untuk Tahun 3 (kecuali 3 item) and Tahun 5 menunjukkan semua item adalah 'fit' dalam julat 0.7 ke 1.3. Kajian ini membincangkan masalah yang telah dihadapi oleh murid semasa menggunakan perwakilan susunan garisan dan dilema yang dihadapi oleh penyelidik semasa melaksanakan kajian. Berdasarkan dapatan kajian ini, terdapat dua implikasi utama iaitu (1) menggunakan perwakilan secara visual dalam pedagogi oleh guru-guru dan (2) faedah-faedah menggunakan model Item-Response Theory (IRT) oleh guru-guru dalam penilaian. Cadangan-cadangan untuk penambahbaikan item dalam ujian, struktur perwakilan susunan garisan dan cara mengutip data yang berlainan juga telah dikemukakan.



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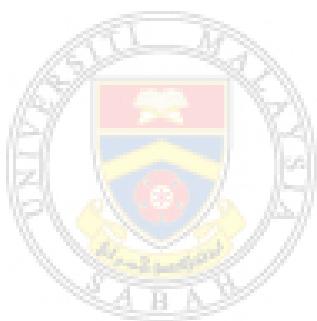
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LIST OF SYMBOLS / ABBREVIATIONS

es	effect size
EPPRD	Education Policy Planning and Research Division
H	Hundreds
HOTS	Higher Order Thinking Skills
Infit MNSQ	Infit Mean Square
IRT	Item-Response Theory
IPM	Item-Person Map
IPMs	Item-Person Maps
IU	Instrumental Understanding
JPNS	Sabah State Education Department
KBSR	Kurikulum Bersepadu Sekolah Rendah
KPM	Kementerian Pendidikan Malaysia
KSSR	Kurikulum Standard Sekolah Rendah
LOTS	Lower Order Thinking Skills
MDM	Multi-Digit Multiplication
MOE	Ministry Of Education
MMR	Mixed Methods Research
Outfit MNSQ	Outfit Mean Square
RU	Relational Understanding
SDM	Single Digit Multiplication
T	Tens
Th	Thousands
U	Units / ones
2D x 1D numbers	multiplication of 2-digit with 1-digit whole numbers
2D x 2D numbers	multiplication of 2-digit with 2-digit whole numbers

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