COEFFICIENT PROBLEMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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SCHOOL OF SCIENCE AND TECHNOLOGY
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ABSTRACT

COEFFICIENT PROBLEMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

This thesis considers A the class of functions which are analytic in the open unit disk $D=\{z:|z|<1\}$. The subclass of A consisting of univalent functions and normalized by the conditions $f(0)=f'(0)-1=0$ is denoted by S. The main subclasses of S includes S^*, C, K and K^* which respectively consists of starlike, convex, close-to-convex and quasi-convex functions. This thesis also considers a subclass of S, denoted by T, consisting of functions f with non negative coefficients. By considering functions $f \in T$, two new subclasses are introduced and properties for functions in these classes are studied such as coefficient estimates, growth and extreme points. In this thesis, the upper bounds for the Fekete-Szegö functional and functional derived from the 2nd Hankel determinant are obtained for certain subclasses of S.

ABSTRAK

Tesis ini mempertimbangkan A *sebagai kelas fungsi yang analisis di dalam cakera unit terbuka* $D = \{z : |z| < 1\}$*. Subkelas bagi* A *yang terdiri daripada fungsi univalen dan ternormal supaya* $f(0) = f'(0) - 1 = 0$ *dilambangkan sebagai* S *. Subkelas utama bagi* S *termasuk* S^*, C, K *and* K^* *masing-masing merupakan fungsi bakbintang, cembung, hampir cembung dan kuasi-cembung. Tesis ini juga mempertimbangkan subkelas bagi* S *, dilambangkan sebagai* T *, terdiri daripada fungsi* f *dengan pekali tak negatif. Dengan mempertimbangkan fungsi* $f \in T$ *, dua subkelas diperkenalkan dan sifat fungsi bagi kedua-dua kelas dikaji seperti anggaran pekali, pertumbuhan dan titik ekstrim. Di dalam tesis ini, batasan atas bagi fungsian Fekete-Szegö and fungsian penentu Hankel ke-2 juga diperoleh bagi subkelas* S *.*

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Chuah Puoi Choo
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$\leq$	less than or equal to
$\geq$	greater than or equal to
$<$	less than
$>$	greater than
$\equiv$	identically equal to
$\in$	an element of
$\subset$	subset of
$\subseteq$	subset of or equal to
$\bar{z}$	conjugate z
$\sum$	summation
$\lvert\ \rvert$	modulus
∞	infinity
$\rightarrow$	approach to
$[a,b]$	closed interval
D	open unit disk
$\bar{D}$	closed disk
E	domain
A	class of analytic functions
S	class of univalent functions and normalized
T	class of analytic functions with negative coefficients
P	class of functions with positive real part
$\mathbb{C}$	complex plane
S^*	class of starlike functions

$S^*(\alpha)$	class of starlike functions of order α
C	class of convex functions
$C(\alpha)$	class of convex functions of order α
K	class of close-to-convex functions
$K(\beta,\alpha)$	class of close-to-convex functions of order β type α
K^*	class of quasi-convex functions
$K^*(\beta,\alpha)$	class of quasi-convex functions of order β type α
R	class of functions with bounded turning
S_s^*	class of starlike functions with respect to symmetric points
$S_s^*(\beta)$	class of starlike functions of order β with respect to symmetric points
S_c^*	class of starlike functions with respect to conjugate points
$S_c^*(\beta)$	class of starlike functions of order β with respect to conjugate points
S_{sc}^*	class of starlike functions with respect to symmetric conjugate points
$S_{sc}^*(\beta)$	class of starlike functions of order β with respect to symmetric conjugate points
C_s	class of convex functions with respect to symmetric points
$C_s(\gamma)$	class of convex functions of order γ with respect to symmetric points
K_s	class of close-to-convex functions with respect to symmetric points
$K_s(\beta,\gamma)$	class of close-to-convex functions of order β type γ with respect to symmetric points

K_s^*	class of quasi-convex functions with respect to symmetric points
$K_s^*(\beta,\gamma)$	class of quasi-convex functions of order β type γ with respect to symmetric points
$R(\alpha)$	class of functions of order α with bounded turning
Max	maximum value
$H_q(n)$	qth Hankel determinant
$S^*T(\alpha)$	class of functions which satisfy $\left|\left(\frac{\alpha z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)-1\right|<\left|\left(\frac{\alpha z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)+1\right|, \quad z\in D$
$CT(\alpha)$	class of functions which satisfy $\left|\left(\frac{\alpha(z^2 f''(z))'}{f'(z)}+\frac{(zf'(z))'}{f'(z)}\right)-1\right|<\left|\left(\frac{\alpha(z^2 f''(z))'}{f'(z)}+\frac{(zf'(z))'}{f'(z)}\right)+1\right|, \quad z\in D$
$S_s^*T(\alpha,\beta)$	class of functions which satisfy $\left|\frac{zf'(z)}{f(z)-f(-z)}-1\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)-f(-z)}+1\right|, \quad z\in D$
$S_c^*T(\alpha,\beta)$	class of functions which satisfy $\left|\frac{zf'(z)}{f(z)+\overline{f(\bar{z})}}-1\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)+\overline{f(\bar{z})}}+1\right|, \quad z\in D$
$S_{sc}^*T(\alpha,\beta)$	class of functions which satisfy $\left|\frac{zf'(z)}{f(z)-\overline{f(-\bar{z})}}-1\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)-\overline{f(-\bar{z})}}+1\right|, \quad z\in D$
$K_s(\alpha,\beta)$	class of functions which satisfy $\operatorname{Re}\left(\frac{2\alpha z^2 f''(z)}{h(z)-h(-z)}+\frac{2zf'(z)}{h(z)-h(-z)}\right)>0, \quad h=zg'\in S_s^*(\beta), \quad z\in D$

$\boldsymbol{K_s^*(\alpha,\beta)}$ class of functions which satisfy

$$\mathrm{Re}\left(\frac{2\alpha z\left(z^2 f''(z)\right)'}{h(z)-h(-z)}+\frac{2z\left(zf'(z)\right)'}{h(z)-h(-z)}\right)>0, \quad h=zg' \in S_s^*(\beta), \quad z\in D$$

$\boldsymbol{K_s(\alpha,\beta,\gamma)}$ class of functions which satisfy

$$\mathrm{Re}\left(\frac{2\alpha z^2 f''(z)}{h(z)-h(-z)}+\frac{2zf'(z)}{h(z)-h(-z)}\right)>\beta, \quad h=zg'(z)\in S_s^*(\gamma), \quad z\in D$$

$\boldsymbol{K_s^*(\alpha,\beta,\gamma)}$ class of functions which satisfy

$$\mathrm{Re}\left(\frac{2\alpha z\left(z^2 f''(z)\right)'}{h(z)-h(-z)}+\frac{2z\left(zf'(z)\right)'}{h(z)-h(-z)}\right)>\beta, \quad h=zg'(z)\in S_s^*(\gamma), \quad z\in D$$

CHAPTER 1

PRELIMINARIES

1.1 Introduction

Geometric function theory is the branch of complex analysis which deals with the geometric properties of analytic functions, founded around the turn of the 20th century. In spite of the famous coefficient problem, the Bieberbach conjecture that was solved by Louis de Branges in 1984 suggests various approaches and directions of studies in the geometric function theory. The cornerstone of geometric function theory is the theory of univalent functions, but new related topics appeared and developed with many interesting results and applications (Bulboacă *et al.*, 2012).

This thesis considers A to be the class of analytic functions in the open unit disk $D=\{z:|z|<1\}$. According to Billing (2010), a function f is said to be analytic at a point z in the domain E if it is differentiable not only at z but also in some neighborhood of point z. A function f is said to be analytic on a domain E if it is analytic at each point of E.

The subclass of A consisting of univalent functions and normalized by the conditions $f(0)=f'(0)-1=0$, is denoted by S. In Goodman (1975), a function f is univalent in domain E if it provides one-to-one mapping onto its image, $f(E)$. In other words, $f(z_1)=f(z_2)$ implies that $z_1=z_2$, for $z_1,z_2\in E$. Univalent functions are also known as '*Schlicht*' which is German word means simple, the Russian refers to such functions as '*odnolistni*' which means single-sheeted.

If $f(z)\in S$ then $f(z)$ has a Maclaurin series expansion of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

where a_n is a complex numbers.

This thesis also considers subclass of S, denoted by T consisting of functions f of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \tag{1.2}$$

where a_n is nonnegative real numbers. The class of T was introduced by Silverman (1975). There are many researchers have studied the class T such as Halim *et al.* (2005), Owa *et al.* (2005), Halim *et al.* (2007), Deng (2007) and Chaurasia and Sharma (2010).

The important subclasses of S include class of starlike functions, S^*, class of convex functions, C, class of close-to-convex functions, K and class of quasi-convex functions, K^* .The definitions of S^*, C, K and K^* will be given in the following section.

1.2 Starlike and Convex Functions

In this section, we give the geometrical representation and analytic description of functions $f \in S^*$.

Definition 1.1 (Goodman, 1975) A set E in the plane is said to be starlike with respect to w_0 an interior point of E if each ray with initial point w_0 intersects the interior of E in a set that is either a line segment or a ray. If a functions $f(z)$ maps D onto a domain which is starlike with respect to w_0, then $f(z)$ is said to be starlike with respect to w_0. In the special case that $w_0=0$, $f(z)$ is a starlike functions.

An example of starlike domain is shown in Figure 1.1 by its geometrical representation.

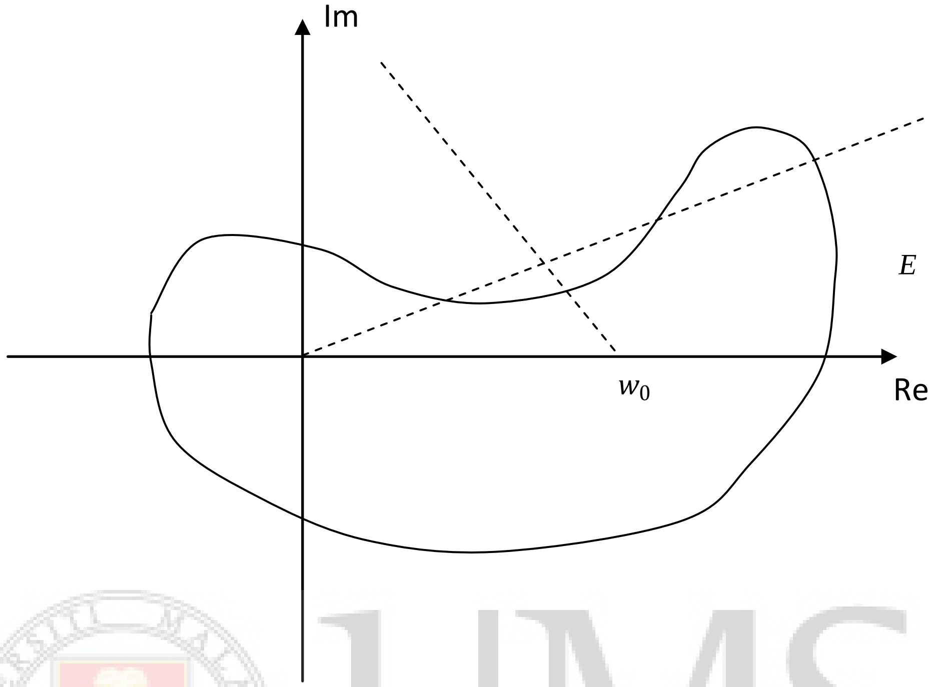

Figure 1.1: Starlike domain

Source: Goodman (1975)

The domain shown in Figure 1.1 is starlike with respect to w_0 but it is not starlike with respect to the origin.

An analytic description of functions $f \in S^*$ is follows.

Theorem 1.1 (Goodman, 1975) Let $f(z)$ be analytic and univalent in the closed disk $\overline{D}: |z| \leq 1$. Then, $f(z)$ maps D onto a region that is starlike with respect to $w = 0$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq 0, \qquad z \in C_R : |z| = 1 \tag{1.3}$$

The Koebe function is a starlike function (Goodman, 1975).

In 1936, Robertson introduced the concept of starlike functions of order α.

Definition 1.2 (Goodman, 1975) A function $f(z)$ of the form (1.1) is said to be starlike of order α in D if for all $z \in D$

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha \tag{1.4}$$

for some $0 \leq \alpha \leq 1$.

The set of all such functions is denoted by $S^*(\alpha)$. According to Owa *et al.* (1986), $S^*(\alpha) \subseteq S^*(0) \equiv S^* \subset S$.

We now give geometrical representation and analytic description of functions $f \in C$.

Definition 1.3 (Goodman, 1975) A set E in the complex plane is called convex if for every pair of points w_1 and w_2 in the interior of E, the line segment joining w_1 and w_2 is also in the interior of E. If a functions $f(z)$ maps D onto a convex domain, then $f(z)$ is called convex functions.

An example of convex domain is shown in Figure 1.2 by its geometrical representation.

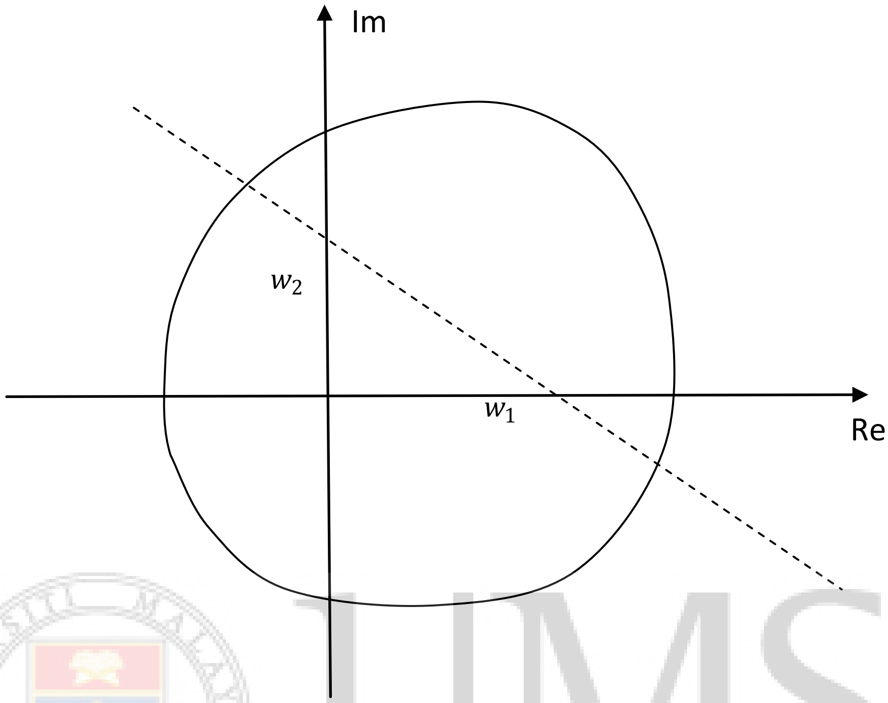

Figure 1.2: Convex domain
Source: Goodman (1975)

An analytic description of functions $f \in C$ is given below.

Theorem 1.2 (Goodman, 1975) Let $f(z)$ be analytic and univalent in the closed disk $\overline{D}:|z|\leq 1$. Then, $f(z)$ maps $\overline{D}$ onto a convex domain if and only if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\geq 0, \qquad z\in C_R:|z|=1 \tag{1.5}$$

The special function $L_0(z)=\dfrac{1+z}{1-z}$ is a convex function because it maps D onto a half-plane.

Alexander (1915) implied that $f \in C$ if and only if $zf'(z)\in S^*$.

Robertson (1936) introduced the concept of convex functions of order α.

Definition 1.4 (Goodman, 1975) A function $f(z)$ of the form (1.1) is said to be convex of order α in D if for all $z \in D$

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq \alpha \qquad (1.6)$$

for some $0 \leq \alpha \leq 1$.

The set of all such functions is denoted by $C(\alpha)$. According to Owa *et al.* (1986), $C(\alpha) \subseteq C(0) \equiv C \subset S$ and $C(\alpha) \subset S^*(\alpha) \subset S$

1.3 Close-to-Convex and Quasi Convex Functions

Another subclass of S is the class of close-to-convex function introduced by Kaplan (1952).

Definition 1.5 (Goodman, 1983) A function $f(z)$ analytic in D is said to be close-to-convex in D (or merely close-to-convex) if there is a function $\phi(z) \in C$ and a real β such that

$$\operatorname{Re}\left(\frac{f'(z)}{e^{i\beta}\phi'(z)}\right) > 0 \qquad (1.7)$$

Let K denote the set of all such functions of the form (1.1) that are close-to-convex in D.

By Alexander's Theorem if $\phi(z) \in C$, then $\Phi = z\phi'(z) \in S^*$. Hence in Definition 1.5 we can replace (1.7) by the condition: there is a $\Phi \in S^*$ is such that

$$\operatorname{Re}\left(\frac{zf'(z)}{e^{i\beta}\Phi(z)}\right) > 0, \qquad z \in D \qquad (1.8)$$

The class of close-to-convex can be generalized to close-to-convex of order β type α as stated in Noor (1987).

Definition 1.6 (Noor, 1987) A function f analytic in D, normalized by the conditions $f(0)=f^{'}(0)-1=0$, is said to be close-to-convex of order β type α where $0 \leq \beta \leq 1$ and $0 \leq \alpha \leq 1$, if and only if there exists a function $g \in S^{*}(\alpha)$ such that, for $z \in D$

$$\operatorname{Re}\left(\frac{zf^{'}(z)}{g(z)}\right) > \beta \tag{1.9}$$

We denote such a class of functions as $K(\beta,\alpha)$. It is clear that $K(0,0)=K$. According to Noor (1987), this class was introduced by Libera in 1964.

In 1980, Noor and Thomas introduced the class quasi-convex which is denoted by K^{*} as given in definition 1.7.

Definition 1.7 (Noor and Thomas, 1980) Let f be analytic in D with $f(0)=f^{'}(0)-1=0$. Then f is said to be quasi-convex in D if there exists a convex functions g with $g(0)=g^{'}(0)-1=0$ such that for $z \in D$,

$$\operatorname{Re}\left(\frac{(zf^{'}(z))^{'}}{g^{'}(z)}\right) > 0, \qquad z \in D \tag{1.10}$$

The class of quasi-convex can be also generalized to quasi-convex of order β type α.

Definition 1.8 (Noor, 1987) A function f analytic in D, normalized by the conditions $f(0)=f^{'}(0)-1=0$, is said to be quasi-convex of order β type α if and only if there exists a function $g \in C(\alpha)$ such that, for $z \in D$

$$\operatorname{Re}\left(\frac{(zf^{'}(z))^{'}}{g^{'}(z)}\right) > \beta \tag{1.11}$$

where $0 \leq \beta \leq 1$ and $0 \leq \alpha \leq 1$.

We denote such class of functions as $K^{*}(\beta,\alpha)$. Clearly $K^{*}(0,0)=K^{*}$.

1.4 Functions with Positive Real Part

Definition 1.9 (Goodman, 1975) The set P is the set of all functions of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (1.12)$$

that are analytic in D, where $c_1, c_2, \cdots$ are complex numbers and such that for z in D,

$$\mathrm{Re}(p(z)) > 0$$

Any function in P is called a function with positive real part in D.

According to Goodman (1975), the Möbius function of the form

$$L_0(z) \equiv \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \cdots = 1 + 2\sum_{n=1}^{\infty} z^n$$

plays a central role in the class P. This function is analytic and univalent in D, and it maps D onto the half plane and shown in Figure 1.3.

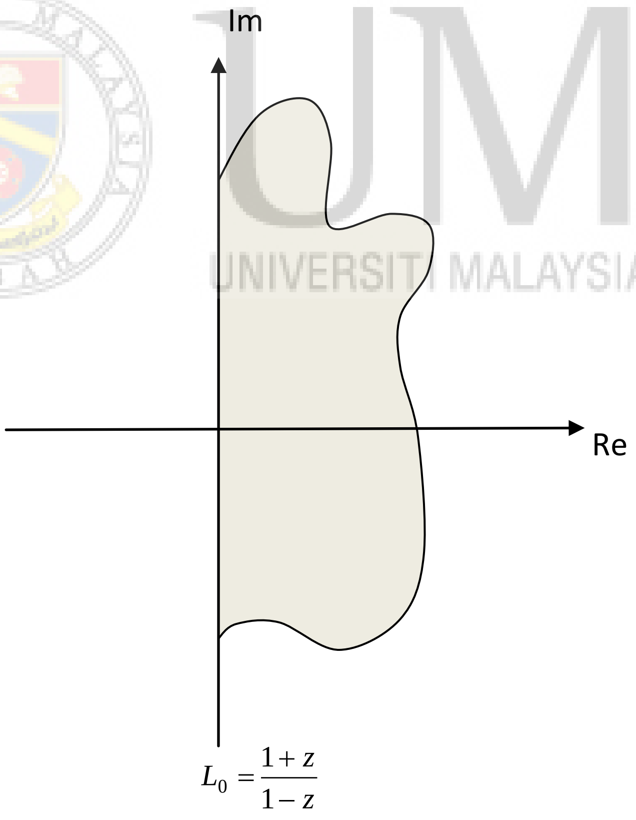

Figure 1.3: Region of half-plane

Source: Goodman (1975)

Here, we state some properties belonging to the class P.

Lemma 1.1 (Cho and Owa, 2003) Let p be analytic in D with $\mathrm{Re}(p(z))>0$ and be given by $p(z)=1+c_1z+c_2z^2+\cdots$ for $z\in D$, then

$$|c_n|\leq 2 \qquad (n\geq 1)$$

and

$$\left|c_2-\frac{c_1^2}{2}\right|\leq 2-\frac{|c_1|^2}{2}$$

Lemma 1.2 (Billing, 2010) Let E be a subset of $\mathbb{C}\times\mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $\Phi:E\to\mathbb{C}$ be a complex function. For $u=u_1+iu_2$, $v=v_1+iv_2$ (u_1,u_2,v_1,v_2 are real), let Φ satisfy the following conditions:

(i) $\Phi(u,v)$ is continuous in E;

(ii) $(1,0)\in E$ and $\mathrm{Re}(\Phi(1,0))>0$; and

(iii) $\mathrm{Re}(\Phi(iu_2,v_1))\leq 0$ for all $(iu_2,v_1)\in E$ and such that $v_1\leq -\frac{(1+u_2^2)}{2}$.

Let $p(z)=1+c_1z+c_2z^2+\cdots$ be analytic in the unit disk D, such that $(p(z),zp'(z))\in E$ for all $z\in D$. If

$$\mathrm{Re}\left(\Phi(p(z),zp'(z))\right)>0, \quad z\in D$$

then $\mathrm{Re}(p(z))>0$ in D.

Lemma 1.3 (Mehrok *et al.*, 2011) If $p\in P$, then

$$2c_2=c_1^2+x\left(4-c_1^2\right) \tag{1.13}$$

and

$$4c_3=c_1^3+2\left(4-c_1^2\right)c_1x-c_1\left(4-c_1^2\right)x^2+2\left(4-c_1^2\right)\left(1-|x|^2\right)z \tag{1.14}$$