COEFFICIENT BOUNDS FOR CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS

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ABSTRACT

This study considers A to be the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$, $a_n \in \mathbb{C}$ which are analytic in the open unit disc $D=\{z:|z|<1\}$. Further, let S the subclass of A consisting of univalent functions and normalized by the conditions $f(0)=f'(0)-1=0$. The main subclasses of S include of S^*, K and C which respectively consists of starlike, convex and close-to-convex functions. It is well known that every function $f \in S$ has an inverse defined by $f^{-1}(f(z))=z\,(z \in D)$ and $f(f^{-1}(w))=w\,(|w|<r_0(f);r_0(f)\geq 1/4)$ where $f^{-1}(w)=w-a_2w^2+(2a_2{}^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots$. A function is said to be bi-univalent in D if both f and f^{-1} are univalent in D. Let Σ denote the class of bi-univalent functions in D. By considering functions $f \in \Sigma$, new subclasses of Σ were proposed and initial coefficients were obtained for these classes. In a meanwhile, the upper bounds for the Fekete-Szegö and second Hankel functional were obtained for certain subclasses of Σ.

ABSTRAK

BATASAN PEKALI BAGI SUATU KELAS FUNGSI BI-UNIVALEN

Kajian ini mempertimbangkan A *sebagai kelas fungsi berbentuk* $f(z)=z+\sum_{n=2}^{\infty}a_n z^n,\ a_n\in\mathbb{C}$ *yang analisis di dalam cakera unit terbuka* $D=\{z:|z|<1\}$ *. Selanjutnya, andaikan* S *sebagai subkelas bagi* A *yang terdiri daripada fungsi univalen dan ternormal dengan keadaan* $f(0)=f'(0)-1=0$ *. Subkelas utama bagi* S *termasuk* S^* *,* K *dan* C *masing-masing mengandungi fungsi bakbintang, cembung dan hampir cembung. Setiap fungsi* $f\in S$ *mempunyai songsangan ditakrifkan sebagai* $f^{-1}(f(z))=z\ (z\in D)$ *dan* $f(f^{-1}(w))=w\ (|w|<r_0(f);r_0(f)\geq 1/4)$ *dengan* $f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots$ *. Suatu fungsi dikatakan bi-univalen di dalam* D *jika* f *dan* f^{-1} *adalah univalen di dalam* D *. Andaikan* Σ *menandakan kelas fungsi bi-univalen di dalam* D *. Dengan mempertimbangkan fungsi* $f\in\Sigma$ *, subkelas baru bagi* Σ *diperkenalkan dan pekali awalan diperoleh bagi fungsi di dalam kelas-kelas ini. Di samping itu, batasan atas bagi fungsian Fekete-Szegö dan penentu Hankel kedua juga diperoleh bagi subkelas* Σ *.*

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LIST OF SYMBOLS

$\leq$	less than or equal to
$<$	less than
$\geq$	greater than or equal to
$>$	greater than
$\lvert \; \rvert$	modulus
$\in$	an element of
Σ	class of bi-univalent functions
∞	infinity
Re	real part of
Δ	domain
$\neq$	not equal to
$\subset$	subset of
D	unit disk
E	set
S	class of univalent functions
S^*	class of starlike functions
K	class of convex functions
C	class of close-to-convex functions
A	class of analytic functions
$SK(\alpha)$	class of strongly convex function of order α
$L_o(z)$	Möbius function
$\mathbb{C}$	complex plane
$\arg$	argument
$\max$	maximum value
$SS^*(\alpha)$	class of strongly starlike functions of order α
$S^*(\alpha)$	class of starlike functions of order α
$K(\alpha)$	class of convex functions of order α
$S^*_\Sigma(\alpha)$	class of bi-starlike functions of order α
$K_\Sigma(\alpha)$	class of bi-convex functions of order α

P	-	set of all functions of the form $p(z)=1+c_1z+c_2z^2+\cdots=1+\sum_{n=1}^{\infty}c_nz^n$
C_s	-	class of convex functions with respect to symmetric points
exp	-	exponential
$H_q(n)$	-	q -th Hankel determinant
H_α	-	class of functions which satisfy $\mathrm{Re}\left(\frac{\alpha z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)>0$
C_α	-	class of functions which satisfy $\mathrm{Re}\left(\frac{\left(\alpha z^2 f''(z)+zf'(z)\right)'}{f'(z)}\right)>0$

CHAPTER 1

PRELIMINARIES

1.1 Introduction

Geometric function theory is one of the aspects of the theory of analytic functions of a complex variable. We recall that a function of the complex variable z is analytic in an open set if it has a derivative at each point in that set. In particular, f is analytic at a point z_0 if it is analytic in a neighborhood of z_0 (Brown and Churchill, 1996). For instance, the function $f(z)=1/z$ is analytic at each nonzero point in the finite plane. In this field, we consider the power series of the form

$$f(z)=b_0+b_1z+b_2z^2+\ldots+b_nz^n+\ldots$$

in the complex variable z that are convergent in a domain Δ. Such a power series interpreted as a mapping of the domain Δ in the z-plane onto some range set F in the w-plane. A nice geometric property from the point of view of conformal mapping possessed by an analytic function $f(z)$ is that of univalence in Δ. (Ahuja, 1986).

1.2 Univalent Functions

According to (Ahuja, 1986), a function $f(z)$ that is analytic in Δ is said to be univalent (schlicht) in Δ if it never takes the same value twice, that is $f(z_1)\neq f(z_2)$ if $z_1\neq z_2$, whenever $z_1, z_2\in\Delta$. The theory of univalent functions is so complicated that certain simplifying assumptions are necessary. First is to take the unit disk $D=\{z:|z|<1\}$ in place of arbitrary domain Δ. Second is to take normalization conditions: $f(0)=0$, $f'(0)=1$. With these assumptions, we can rewrite $f(z)$ in the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ a_n\in\mathbb{C} \text{ and } z\in D. \tag{1.1}$$

Further, let

$$A=\{f: f \text{ is analytic and normalized in } D\}$$

and

$$S=\{f: f\in A \text{ and } f \text{ is univalent in } D\}.$$

The function $g(z)=\dfrac{1+z}{1-z}$ is univalent in D, and $g(D)$ is the half plane $\operatorname{Re}(g(z))>0$. The function $k(z)=\dfrac{1}{4}\left[\left(\dfrac{1+z}{1-z}\right)^2-1\right]=\dfrac{z}{(1-z)^2}$ is also belongs to S. This is called the Koebe function and it maps D onto the entire complex plane except the slit along the negative real axis from $-\infty$ to $-1/4$.

1.3 Subclasses of S

We give below some of the most important subclasses of S.

Definition 1.1 (Goodman, 1975) A set E in the plane is said to be starlike with respect to w_o an interior point of E if each ray with initial point w_o intersects the interior of E in a set that is either a line segment or a ray. If a function $f(z)$ maps D onto a domain that is starlike with respect to w_o then we say that $f(z)$ is starlike with respect to w_o. In the special case that $w_o=0$, we say that $f(z)$ is a starlike function.

The class of all starlike functions in D is denoted by S^*. Robertson (1936a) showed that $f\in S^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0, \quad z\in D.$$

According to (Ahuja, 1986), the class S^* was first treated by Alexander in 1916. The Koebe function $K(z)=\frac{z}{(1-z)^2}$ is a starlike function because it maps D onto the entire complex plane minus the slit $-\infty < w \le -\frac{1}{4}$.

Another important subclass of S consists of the convex functions.

Definition 1.2 (Goodman, 1975) A set E in the plane is said to be convex if for every pair of points w_1 and w_2 in the interior E, the line segment joining w_1 and w_2 is also in the interior of E. If a function $f(z)$ maps D onto a convex domain, then $f(z)$ is called convex function.

The class of all convex functions in D is denoted by K. Robertson (1936b) observed that function $f \in K$ if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0, \quad z \in D.$$

According to (Ahuja, 1986), the class K was first studied by Study in 1913. The Möbius function $L_o(z)=\frac{1+z}{1-z}=1+2\sum_{n=1}^{\infty} z^n$ is convex function become it maps D onto a half plane.

In 1952, (Kaplan, 1952) introduced the class C of all close-to-convex functions in Δ.

Definition 1.3 (Kaplan, 1952) Let $f(z)$ be analytic for $|z|<\mathbb{R}$. Then $f(z)$ is close-to-convex for $|z|<\mathbb{R}$ if there exists a function $\varphi(z)$, convex and schlicht for $|z|<\mathbb{R}$, such that $\frac{f'(z)}{\varphi'(z)}$ has positive real part for $|z|<\mathbb{R}$.

Kaplan (1952) proved that $C \subset S$. From the definitions of the above subclasses of S, it follows that $K \subset S^* \subset C \subset S$.

The classes S^* and K can be generalised to the class of the starlike and convex functions of order α as follows.

Definition 1.4 (Goodman, 1975) A function $f(z)$ of the form (1.1) is said to be starlike of order α in D if for all $z \in D$

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in D)$$

where $(0 \le \alpha < 1)$. We denote $S^*(\alpha)$ the subclass of A consisting of all starlike functions of order α in D.

Note that a function $f \in A$ is said to be starlike in D when $\alpha = 0$.

Definition 1.5 (Goodman, 1975) A function $f(z)$ of the form (1.1) is said to be convex of order α in D if for all $z \in D$

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in D)$$

where $(0 \le \alpha < 1)$. We denote $K(\alpha)$ the subclass of A consisting of all convex functions of order α in D.

Note that a function $f \in A$ is said to be convex in D when $\alpha = 0$.

The concepts of starlike and convex functions of order α were introduced by (Robertson, 1936b). After that, many of the mathematicians continue to explore this idea, by interesting way, for example (Brannan and Kirwan, 1969). One can alter the condition $\operatorname{Re}(Q(z))>0$ in a different, but equally interesting way, by demanding that $|\arg Q(z)|<\alpha\pi/2$.

Definition 1.6 (Goodman, 1975) A function f of the form (1.1) is said to be strongly starlike of order α in D if for all z in D,

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2},\quad (0<\alpha\leq 1).$$

The set of all such functions is denoted by $SS^*(\alpha)$.

Definition 1.7 (Goodman, 1975) A function f of the form (1.1) is said to be strongly convex of order α in D if for all z in D,

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\alpha\pi}{2},\quad (0<\alpha\leq 1).$$

The set of all such functions is denoted by $SK(\alpha)$.

1.4 Bi-univalent Functions

The Koebe one-quarter theorem (Duren,1983) ensures that the image of D under every $f\in S$ contains a disk of radius $\frac{1}{4}$. Therefore, every $f\in S$ has an inverse function f^{-1} satisfying $f^{-1}(f(z))=z\ (z\in D)$ and

$$f^{-1}(f(w))=w\quad\left(|w|<r_0(f);r_0(f)\geq\frac{1}{4}\right)$$

where

$$f^{-1}(w)=w-a_2w^2+\left(2a_2^{\,2}-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\dots. \tag{1.2}$$

Thus, a function $f \in A$ is said to be bi-univalent in D if both f and f^{-1} are univalent in D. Let Σ denote the class of bi-univalent functions in D of the functions of the form (1.1). Examples of functions in the class Σ are

$$f(z)=\frac{z}{1-z},\ f(z)=\log\left(\frac{1}{1-z}\right),\ f(z)=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

However, the familiar Koebe function is not a member of Σ. Other common examples of functions

$$f(z)=\frac{2z-z^2}{2} \text{ and } f(z)=\frac{z}{1-z^2}$$

are not members of Σ.

Lewin (1967) investigated the class Σ and showed that $|a_2|<1.51$. Afterwards, (Brannan and Clunie, 1979) conjectured that $|a_2|\le\sqrt{2}$. Netanyahu (1969), on the other hand, showed that $\max_{f\in\Sigma}|a_2|=\frac{4}{3}$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad \left(n\in\mathbb{N}\setminus\{1,2\};\mathbb{N}:=\{1,2,3,\ldots\}\right)$$

is presumably still an open problem.

(Brannan and Taha, 1986) and (Taha, 1981) introduced certain subclasses of Σ called bi-starlike and bi-convex functions which similar to the familiar subclasses of S stated in Definition (1.1) and Definition (1.2). Furthermore, following (Brannan and Taha, 1986) and (Taha, 1981), a function $f \in A$ is in the class $SS_{\Sigma}^{*}(\alpha)$ ($0<\alpha\le 1$) of strongly bi-starlike functions of order α if each of the following conditions are satisfied:

$$f\in\Sigma \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2} \quad (z\in D;0<\alpha\le 1)$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2} \quad (w\in D;0<\alpha\le 1),$$

where g is the extension of f^{-1} to D. The classes $S_{\Sigma}^{*}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the classes $S^{*}(\alpha)$ and $K(\alpha)$ defined by Definition (1.4) and Definition (1.5), were also introduced analogously. (Brannan and Taha, 1986) and (Taha, 1981) found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for each of the classes $S_{\Sigma}^{*}(\alpha)$ and $K_{\Sigma}(\alpha)$.

1.5 Functions with Positive Real Part

Next, we give the definition of function with positive real part which is closely related to S^{*} and K.

Definition 1.8 (Goodman, 1975) The set P is the set of all functions of the form

$$p(z)=1+c_1z+c_2z^2+\cdots=1+\sum_{n=1}^{\infty}c_nz^n \tag{1.3}$$

that are analytic in D where c_n is a complex number and such that for z in D, $\operatorname{Re}(p(z))>0$. Any function in P is called a function with positive real part in D.

According to (Duren, 1983), one of the example function in class P is Möbius function

$$L_0(z)=\frac{1+z}{1-z}=1+2z+2z^2+\cdots=1+2\sum_{n=1}^{\infty}z^n.$$

This function is in the class P, it is analytic and univalent in D and maps D onto the half-plane.

1.6 Objectives of Study

The objectives of this study are:

i. to propose certain subclasses of Σ and determine the initial coefficients of functions in these subclasses;

ii. to determine the upper bounds of the Fekete-Szegö functional for functions in certain subclasses of Σ; and

iii. to determine the upper bounds of second Hankel determinant for functions in certain subclasses of Σ.

1.7 Thesis Outline

There are 5 chapters in this study. Chapter 1 gives an overview of the ideas of geometric function theory, analytic and univalent functions, some definitions that will be referred throughout this study. Chapter 2 giving certain subclasses of bi-univalent functions which are denoted by $A_\Sigma(\alpha,\lambda)$, $A_\Sigma(\beta,\lambda)$, $B_\Sigma(\alpha,\lambda)$, $B_\Sigma(\beta,\lambda)$, $D_\Sigma(\alpha,\lambda)$, $D_\Sigma(\beta,\lambda)$, $E_\Sigma(\alpha,\lambda)$ and $E_\Sigma(\beta,\lambda)$ and obtaining the initial coefficients of function f in the classes $A_\Sigma(\alpha,\lambda)$, $A_\Sigma(\beta,\lambda)$, $B_\Sigma(\alpha,\lambda)$, $B_\Sigma(\beta,\lambda)$, $D_\Sigma(\alpha,\lambda)$, $D_\Sigma(\beta,\lambda)$, $E_\Sigma(\alpha,\lambda)$ and $E_\Sigma(\beta,\lambda)$. Chapter 3 giving the upper bounds of Fekete-Szegö functional for certain subclasses of Σ which are denoted by $A_\Sigma(\alpha,\lambda)$, $A_\Sigma(\beta,\lambda)$, $B_\Sigma(\alpha,\lambda)$, $B_\Sigma(\beta,\lambda)$, $D_\Sigma(\alpha,\lambda)$, $D_\Sigma(\beta,\lambda)$, $E_\Sigma(\alpha,\lambda)$ and $E_\Sigma(\beta,\lambda)$. Next, Chapter 4 giving the results on the upper bounds of the second Hankel determinant for certain subclasses of Σ which are denoted by $A_\Sigma(\beta,\lambda)$ and $B_\Sigma(\beta,\lambda)$. Finally, Chapter 5 giving the conclusion and future works of this study.

CHAPTER 2

INITIAL COEFFICIENTS

2.1 Introduction

In Chapter 1, brief histories of functions which are in class Σ have been described. This chapter develops new subclasses of Σ which are denoted by $A_\Sigma(\alpha,\lambda)$, $A_\Sigma(\beta,\lambda)$, $B_\Sigma(\alpha,\lambda)$, $B_\Sigma(\beta,\lambda)$, $D_\Sigma(\alpha,\lambda)$, $D_\Sigma(\beta,\lambda)$, $E_\Sigma(\alpha,\lambda)$ and $E_\Sigma(\beta,\lambda)$ with $0<\alpha\le 1$, $0\le\beta<1$ and $\lambda\ge 0$. The ideas of developing new subclasses of Σ was inspired by (Ramesha, Kumar and Padmanabhan, 1995), (Shanmugam, Stephen and Subramanian, 2012) and (Sakaguchi, 1959). Some initial coefficients of functions belonging to the classes $A_\Sigma(\alpha,\lambda)$, $A_\Sigma(\beta,\lambda)$, $B_\Sigma(\alpha,\lambda)$, $B_\Sigma(\beta,\lambda)$, $D_\Sigma(\alpha,\lambda)$, $D_\Sigma(\beta,\lambda)$, $E_\Sigma(\alpha,\lambda)$ and $E_\Sigma(\beta,\lambda)$ are obtained.

2.2 Classes $A_\Sigma(\alpha,\lambda)$ and $A_\Sigma(\beta,\lambda)$

Before giving the definitions of the new subclasses of Σ, we begin by stating the known definitions of classes which were introduced by (Ramesha *et al.*, 1995) and (Shanmugam *et al.*, 2012).

Definition 2.1 (Ramesha *et al.*, 1995) Let f be analytic functions in D with the condition $f(0)=f'(0)-1=0$. Then f is in class H_α if and only if

$$\mathrm{Re}\left(\frac{\alpha z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)>0 \tag{2.1}$$

for $\alpha\ge 0$ and $\frac{f(z)}{z}\ne 0$, $z\in D$.

Definition 2.2 (Shanmugam *et al.*, 2012) Let f be given by (1.1). Then $f \in C_\alpha$ if and only if

$$\operatorname{Re}\left(\frac{\left(zf'(z)+\alpha z^2 f''(z)\right)'}{f'(z)}\right) > 0 \ . \qquad (2.2)$$

Motivated by the class H_α, we come out with the subclasses of Σ which are denoted by $A_\Sigma(\alpha,\lambda)$ and $A_\Sigma(\beta,\lambda)$ as defined below.

Definition 2.3 A function $f(z) \in \Sigma$ given by (1.1) is said to be in class $A_\Sigma(\alpha,\lambda)$ with $0 < \alpha \le 1$ and $\lambda \ge 0$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left(\frac{\lambda z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad z \in D \qquad (2.3)$$

and

$$\left|\arg\left(\frac{\lambda w^2 g''(w)}{g(w)} + \frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}, \quad w \in D \qquad (2.4)$$

and the function g is given by

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \dots \qquad (2.5)$$

Definition 2.4 A function $f(z) \in \Sigma$ given by (1.1) is said to be in class $A_\Sigma(\beta,\lambda)$ with $0 \le \beta < 1$ and $\lambda \ge 0$ if the following conditions are satisfied:

$$\operatorname{Re}\left(\frac{\lambda z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > \beta \qquad (2.6)$$

and

$$\operatorname{Re}\left(\frac{\lambda w^2 g''(w)}{g(w)} + \frac{wg'(w)}{g(w)}\right) > \beta \qquad (2.7)$$

and the function g is given by equation (2.5).

In particular, for $\lambda = 0$ the classes $A_\Sigma(\alpha,\lambda)$ and $A_\Sigma(\alpha,\beta)$ become the class $SS^*_\Sigma(\alpha)$ of strongly bi-starlike functions of order α and the class $S^*_\Sigma(\beta)$ of bi-starlike functions of order β, respectively.

In order to obtain the initial coefficients, we need the following lemma.

Lemma 2.1 (Pommerenke, 1975) If $p \in P$ then $|p_k| \le 2$ for each k, where P is the family of all functions p in D for which $\mathrm{Re}(p(z)) > 0$, $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$ for $z \in D$.

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $A_\Sigma(\alpha,\lambda)$.

Theorem 2.1 Let f given by (1.1) be in the class $A_\Sigma(\alpha,\lambda)$ where $0 < \alpha \le 1$ and $\lambda \ge 0$. Then

$$|a_2| \le \frac{2\alpha}{\sqrt{4\lambda\left[\lambda(1-\alpha)+\alpha+1\right]+\alpha+1}} \tag{2.8}$$

and

$$|a_3| \le \frac{4\alpha^2}{(2\lambda+1)^2} + \frac{\alpha}{3\lambda+1}. \tag{2.9}$$

Proof. Since $f \in A_\Sigma(\alpha,\lambda)$, it follows from (2.3) and (2.4) that

$$\frac{\lambda z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} = [p(z)]^\alpha \tag{2.10}$$

and

$$\frac{\lambda w^2 g''(w)}{g(w)} + \frac{wg'(w)}{g(w)} = [q(w)]^\alpha \tag{2.11}$$

where $p(z)$ and $q(w)$ in P have the forms $p(z)=1+p_1z+p_2z^2+\dots$ and $q(w)=1+q_1w+q_2w^2+\dots$ respectively.

From (1.1), we have

$$f(z)=z+a_2z^2+a_3z^3+\dots.$$

Thus, by differentiating function $f(z)$ with respect to z, we obtain

$$f'(z)=1+2a_2z+3a_3z^2+\dots \tag{2.12}$$

and

$$f''(z)=2a_2+6a_3z+\dots \tag{2.13}$$

Next, from (2.10) we obtain

$$\begin{aligned}&\lambda z^2f''(z)+zf'(z)\\&=\lambda z^2(2a_2+6a_3z+\dots)+z(1+2a_2z+3a_3z^2+\dots)\\&=2\lambda a_2z^2+6\lambda a_3z^3+\dots+z+2a_2z^2+3a_3z^3+\dots\\&=z+2a_2(\lambda+1)z^2+3a_3(2\lambda+1)z^3+\dots\end{aligned}$$

Subsequently, by using a long division method, yields

$$\frac{\lambda z^2f''(z)+zf'(z)}{f(z)}=1+a_2(2\lambda+1)z+\left[2a_3(3\lambda+1)-a_2^2(2\lambda+1)\right]z^2+\dots.$$

By using the similar approach to (2.7), we have

$$g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-\dots.$$

By differentiating function $g(w)$ with respect to w, we get

$$g'(w)=1-2a_2w+3(2a_2^2-a_3)w^2-\dots \tag{2.14}$$

and

$$g''(w)=-2a_2+6(2a_2^2-a_3)w-\dots \tag{2.15}$$

Thus, from (2.11) we get

$$\begin{aligned}
&\lambda w^2 g''(w) + wg'(w) \\
&= \lambda w^2\left[-2a_2 + 6\left(2a_2^{\,2} - a_3\right)w + \dots\right] + w\left[1 - 2a_2 w + 3\left(2a_2^{\,2} - a_3\right)w^2 + \dots\right] \\
&= w - 2a_2(\lambda+1)w^2 + 3\left(2a_2^{\,2} - a_3\right)(2\lambda+1)w^3 + \dots
\end{aligned}$$

Again, by using a long division method, yields

$$\frac{\lambda w^2 g''(w) + wg'(w)}{g(w)} = 1 - a_2(2\lambda+1)w + \left[2a_3\left(2a_2^2 - a_3\right)(3\lambda+1) - a_2^2(2\lambda+1)\right]w^2 + \dots$$

Hence, equation (2.10) gives

$$\begin{aligned}
&1 + a_2(2\lambda+1)z + \left[2a_3(3\lambda+1) - a_2^2(2\lambda+1)\right]z^2 + \dots \\
&= 1 + \alpha p_1 z + \left[\alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^{\,2}\right]z^2 + \dots
\end{aligned} \tag{2.16}$$

and equation (2.11) gives

$$\begin{aligned}
&1 - a_2(2\lambda+1)w + \left[2a_3\left(2a_2^2 - a_3\right)(3\lambda+1) - a_2^2(2\lambda+1)\right]w^2 + \dots \\
&= 1 + \alpha q_1 w + \left[\alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^{\,2}\right]w^2 + \dots
\end{aligned} \tag{2.17}$$

Next, by suitably comparing coefficients of z and z^2 in (2.16) and comparing coefficients of w and w^2 in (2.17), we get

$$a_2(2\lambda+1) = \alpha p_1\,, \tag{2.18}$$

$$2a_3(3\lambda+1) - a_2^{\,2}(2\lambda+1) = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^{\,2}\,, \tag{2.19}$$

$$-a_2(2\lambda+1) = \alpha q_1\,, \tag{2.20}$$

$$2\left(2a_2^{\,2} - a_3\right)(3\lambda+1) - a_2^{\,2}(2\lambda+1) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^{\,2}\,. \tag{2.21}$$

By dividing (2.18) and (2.20), we get

$$p_1 = -q_1\,.$$