

APPROXIMATION FOR PI USING SIMPLE CONTINUED FRACTION

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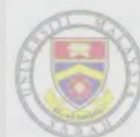
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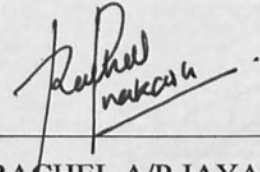
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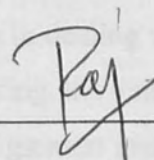


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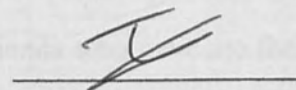
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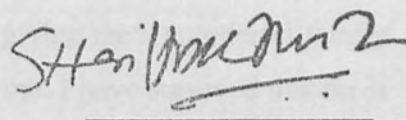
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ABSTRACT

Pi, denoted as π , is a famous mathematical constant and it existed more than 4000 years ago. It is an irrational number which was proven by the Alsatian mathematician and it is also transcendental which was proven by Johann Heinrich Lambert. The main objective of this dissertation is to get an approximation for π by using Microsoft Excel 2003 software and also to learn and understand simple continued fraction. The value of π used for the computation is $\pi = 3.14159265358979$. Simple continued fraction is the method used to solve this problem and it is therefore explained in detail in this dissertation. In the process of getting a good approximation of an irrational number such as π , the convergents of the simple continued fraction of π is a good approximation. The results shows 165 approximations of π by using the value $\pi = 3.14159265358979$. Hence by comparing the convergents from level 0 to level 164, the last convergent which is

C_{164} is the best approximation with the value $\frac{1.38305662115529 \times 10^{84}}{4.40240595665677 \times 10^{83}}$.

PENGANGGARAN UNTUK π DENGAN MENGGUNAKAN PECAHAN BERLANJAR

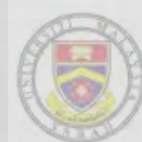
ABSTRAK

Pi, yang mempunyai simbol π , merupakan pemalar dalam bidang matematik yang terkemuka dan ia wujud lebih daripada 4000 tahun dahulu. Ia merupakan nombor bukan nisbah yang dibuktikan oleh ahli matematik Alsatian dan juga ‘transcendental yang dibuktikan oleh Johann Heinrich Lambert’. Objektif utama dalam disertasi ini adalah untuk mendapatkan anggaran yang terbaik bagi π dengan menggunakan perisian Microsoft Excel 2003 serta mempelajari dan memahami konsep pecahan berlanjar. Nombor yang digunakan untuk pengiraan adalah $\pi = 3.14159265358979$. Kaedah yang digunakan untuk menyelesaikan masalah ini ialah pecahan berlanjar dan ia dijelaskan dengan terperinci dalam disertasi ini. Dalam proses mendapatkan anggaran yang terbaik bagi nombor bukan nisbah seperti π , konvergen yang diperolehi daripada pecahan berlanjar merupakan anggaran yang baik. Keputusan menunjukkan bahawa terdapat 165 anggaran bagi π dengan menggunakan nilai $\pi = 3.14159265358979$. Oleh itu, dengan membandingkan kesemua konvergen yang diperolehi dari peringkat 0 hingga peringkat 164, konvergen yang terakhir iaitu C_{164} adalah anggaran yang

terbaik iaitu $\frac{1.38305662115529 \times 10^{84}}{4.40240595665677 \times 10^{83}}$.

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	Real positive numbers	
	Equal sign	
	Approximate equal sign	
()	Curly braces	
[]	Parentheses	
{ }	Square bracket	
<	Less than sign	
>	Greater than sign	
≤	Less than or equal to sign	
≥	Greater than or equal to sign	
≠	Not equal to sign	
∴	Therefore, it follows that	
∝	Proportionality sign	
+	Addition sign	
-	Subtraction sign	
×	Multiplication sign	
÷	Division sign	
∑	Summation sign	
√	Square root sign	
∫	Integral sign	



LIST OF SYMBOLS

π	Pi
\mathbb{Z}	Integers
\mathbb{N}	Natural numbers
\mathbb{R}	Real numbers
\mathbb{R}^+	Real positive numbers
$=$	Equal sign
\approx	Approximate equal sign
$\{ \}$	Curly bracket
$()$	Parentheses
$[]$	Square bracket
\leq	Inequality sign
\geq	Inequality sign
\langle	Inequality sign
\rangle	Inequality sign
\in	Element of
\therefore	Therefore, it follows that
\cdot	Multiplication sign
$+$	Addition sign
$-$	Subtraction sign
$/$	Division sign
\sum	Summation sign
$\sqrt{\quad}$	Square root sign
\int	Integral sign



CHAPTER 1

INTRODUCTION

1.1 Introduction

A mathematical constant is usually a real number or a complex number that appears naturally in mathematics and its value does not change. Mathematical constants are defined independently of any physical measurement. Many particular numbers have special significance in mathematics, and arise in many different contexts. Mathematical constants are elements of the real field of complex numbers. Pi, denoted π , is a famous mathematical constant and it existed long time ago. A research on the approximation of π will be done in this dissertation.

The letter π is a name of a greek letter and the symbol π is invented by William Jones (1675-1749) and is always pronounced like “pie” in English. According to Figure 1.1, π is a name given to the ratio of the circumference, C or the parameter of a circle to the diameter, $d = 2r$.

$$\pi = \frac{C}{d} = \frac{C}{2r}$$

That means, for any circle, you can divide the circumference which is the distance around the circle by the diameter and always get exactly the same number.

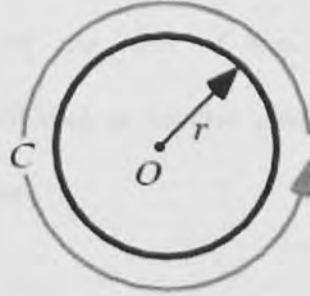


Figure 1.1 Circumference and the diameter of the circle

The constant π is an irrational number; that is it cannot be written as the ratio of two integers and this was proven for the first time by the Alsatian mathematician, Johann Heinrich Lambert in 1766. Apart from that, it is also known that π is transcendental. A famous, but extremely complicated proof of this was provided by the Munich mathematician Ferdinand Lindemann in 1882. This states that there is no polynomial with rational coefficients of which π is a root.

One of the oldest challenges which mathematicians have faced is to assume to be the mathematical analysis of the circle. Statements as to how the circumference of the circle or the area of the circle can be expressed through other variables are found in the oldest mathematical documents. In the beginning, $\pi = 3$. This value was sufficient for centuries for all practical applications, for example in surveying, astronomy and architecture. When other approximations which are better than 3 were discovered, that is where the real history of π begins. By the start of the second millennium BC, such approximations had been developed in several places. Since then, 4000 years of research into π have taken place.

1.2 History of π

When other approximations which is better than 3 were discovered, that is where the real history of π begins. By the start of the second millenium BC, such approximations had been developed in several places. Since then, 4000 years of research into π have taken place.

a. Babylon

According to the ancient history, a clay tablet unearthed in 1936 from the Old Babylon period, approximately 1900-1600 BC, states that the circumference of an

hexagon is $0;57,36$ (in base 60) $= \frac{96}{100} = \frac{24}{25}$ times the circumference of the

circumscribed circle (Katz, 1993). From $u_{hexagon} = 3 \cdot d = \frac{24}{25} \cdot u_{circle} = \frac{24}{25} \cdot \pi \cdot d$ we

get what is perhaps to be the oldest approximation to π ,

$$\pi_{Babylon} = 3\frac{1}{8} = 3.125 = \pi - 0.0165\dots$$

b. Eygpt

In ancient times, it was not absolutely necessary to calculate the circumference of a circle. It is actually quite tedious and difficult to reach at an accurate figure for the area of circle by measurement, so it was highly desirable to be able to calculate it.

Problem no.50 in the Egyptian *Rhind Papyrus* (named after A.H. Rhind, a Scot who purchased it in 1858 in Luxor), which dates back to around 1850 BC, reads as follows

(Katz, 1993): “Example of a round field of diameter 9. What is the area? Take away $\frac{1}{9}$ of the diameter; the remainder is 8. Multiply 8 times 8; it makes 64. Therefore, the area is 64.” The Egyptian scholar was using the formula $A = (d - d/9)^2 = (8d/9)^2$. When the formula stated is compared with the formula for the area, $A = \pi \cdot d^2 / 4$, we get the approximation

$$\pi_{\text{Egypt}} = \left(\frac{16}{9}\right)^2 = 3.16049\dots = \pi + 0.0189\dots$$

c. India

In India, the Indian *Sulvasutras* which means “cord rules”, i.e. rules for building altars of certain shapes with the aid of pieces of cord (Ebbinghaus & Remmert, 1990), which is certainly older than the surviving version dating from 600 BC (Katz, 1993), also describes a calculation of the area of a circle: “If you wish to turn a circle into a square, divide the diameter into 8 parts, and again one of these 8 parts into 29 parts; of these 29 parts remove 28, and moreover the sixth part (of the one left) less the eighth part (of the sixth part).” This however results in a side length s of the required square of

$$s = d \frac{1}{8} \left(7 + \frac{1}{29} \left(1 - \frac{1}{6} \left(1 - \frac{1}{8} \right) \right) \right)$$

$$s = d \frac{9785}{11136}$$

and hence

$$\pi_{\text{India}} = 4s^2 / d^2 = \left(\frac{9785}{5568}\right)^2 = 3.08832\dots = \pi - 0.0532\dots$$

Another famous approximation to π which has been written by the Chinese may actually also be of Indian origin. Which is $\pi \approx \sqrt{10} = 3.16227\dots$

Tropfke referred to a source to which $\sqrt{10}$ is already found in the year 150 BC in the writings of the Indian scholar Umasvati. The author apparently even believes that he can date this value back to 500 BC in India (Tropfke, 1940).

d. Bible

The Bible also mentions a value which is an approximation for π . An architect named Hiram of Tyre was commissioned by King Solomon to build a circular water reservoir out of ore. In 1 Kings 7:23 and also in 2 Chronicles 4:2, says that: “And he made a molten sea of ten cubits from brim to brim...., and a line of 30 cubits did compass it round about.” This shows that $\pi_{\text{Bible}} = 3$. However during that time, more accurate values of π had been known. Many people have ridiculed the Bible for being unsuccessful to come up with a better approximation. For example in the 18th century, it was declared that the molten sea must have been a hexagon (Tropfke, 1940). One example here is that Stern, who concludes from the difference between the spoken and the written versions of the Bible that the actual value of π is obtained by division of both values (Stern, 1985). Giving

$$\pi_{\text{Bible}} = \frac{333}{106} = 3.141509\dots = \pi - 0.000083\dots$$

e. Ancient Greece

Most of the great Greek mathematicians from the 5th to the 3rd centuries BC worked on problems relating to circles. One important discovery of π to the mathematicians was the method of exhaustion. In this method, which is attributed to Antiphon (c.430 BC) or Eudoxos (408-355 BC), the area of a two dimensional figure like the circle can be arrived at through mental “exhaustion” using ever more elaborate versions of figures whose areas are known, like the polygon. However, none of the Greek mathematicians have proven the numeric value of π . Nevertheless the philosopher Plato (427-348 BC) is supposed to have obtained a very accurate value for π during his days, which is $\sqrt{2} + \sqrt{3} = 3.146 \dots$ (Ebbinghaus & Remmert, 1990). This expressions shows that Plato probably arrived at the formula as the arithmetical mean of the half-perimeters of the inscribed square $2\sqrt{2}$ and the circumscribed hexagon $2\sqrt{3}$. These initial values with 3.464... and 2.828..., are weak approximation for π , and yet the end value is almost accurate.

1.3 Research Objectives

The objectives of this research are:

- a. To evaluate the approximation of π by using simple continued fraction.
- b. To learn and understand simple continued fraction.
- c. To calculate the approximation of π by using simple continued fraction with Microsoft Excel program.

1.4 Research Scope

This dissertation will mainly revolve around using simple continued fraction to approximate the value of π . Since π is irrational, therefore only 14 decimal places of the original value of π which have been discovered will be used. Therefore, the convergents of π will be found by using simple continued fraction with the help of Microsoft Excel software.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The existence of the number π has fascinated many people around the world especially today. There are a surprising amount of the most important mathematics and a significant amount of the most important mathematicians who have contributed to the unfolding of π , directly or otherwise (Beggren *et al.*, 1997).

π is one of the few basic principles in mathematics that produces a response of recognition and interest in those who is not concerned professionally with the subject. It has been a part of human culture and the educated imagination for more than 2500 years. The computation of π is the only topic from the most ancient history of mathematics that is still in interest to the modern mathematical research. In order to pursue this topic about π as it developes throughout the years is to follow the path through the history of mathematics that is related to geometry, analysis and special functions, numerical analysis, algebra and number theory. It offers a subject which

provides mathematicians with examples of many current mathematical techniques as well as their historical development.

2.2 Calculation of π

The history of the calculation of π may be broken down into three distinct periods.

2.2.1 The First Period

The first period or the so-called geometrical period, extended from the earliest empirical determination of the ratio of the circumference of a circle to its diameter to the invention of the calculus about the middle of the 17 century. The main effort was directed towards the approximation of this ratio by the calculation of perimeters or areas of regular inscribed and circumscribed polygons.

a. Greece

It began around 250 B.C. with the Greek mathematician Archimedes of Syracuse. He provided the first major landmark in the quest for digits of π . He systematically approximated the number π and produced a upper and lower limits on its value. According to his book *The Measurement of the Circle*, Archimedes (250 B.C.) stated three teorems about the circle and the third of which goes as follows:

3. The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.

When expressed as a formula, the theorem states

$$3 + \frac{1}{7} > \pi > 3 + \frac{10}{71}$$

and this puts π between the limits 3.14084.... and 3.14285..., which are accurate to two decimal places. This marks the origin of the value $\pi = 3\frac{1}{7}$, which spread victoriously from country to country and from book to book (Tropfke, 1940). The method of calculation involves determining all the circumscribed and then all the inscribed polygons of 96 sides. However, Archimedes's polygon method does not converge well enough. The absolute error of the π approximation achieved always remained 0.00112... or -0.00056....

400 years later, the Greek astronomer Ptolemy (150 A.D.) also came up with the approximation 3.1416. He discovered this approximation by simply deriving the

approximate value $3^{\circ}8'30'' = 3\frac{17}{120}$ from the sexagesimal representation of

Archimedes's limits, $3\frac{1}{7} = 3^{\circ}8'34.28''$ and $3\frac{10}{71} = 3^{\circ}8'27.04''$ (Tropfke, 1940), or he

may have separately summed the numerators and denominators from their fractions

$3 + \frac{1}{7} = \frac{154}{49}$ and $3 + \frac{10}{71} = \frac{223}{71}$ to arrive at the mean value of $\frac{377}{120}$ (Ebbinghaus &

Remmert, 1990).

b. China

In China, Liu Xin, an astronomer and calendar expert was asked by Wang Mang at the end of the Western Han dynasty (206 B.C.-24 A.D.) to develop a standard measurement for his empire. Therefore, Liu Xin produce a cylindrical vessel made of bronze. From analysing it, historians have concluded that Liu Xin must have possessed a π approximation which was significantly better than 3 and may even have been the accurate value of 3.1547. Liu Hui (263 A.D.) was one of the scholar whose π calculations were systematic and produce better results. He started out with a circle of radius 10 and by using Pythagoras's theorem, he calculated the areas of inscribed polygons starting with the hexagon and proceeding upwards to polygons with 192 sides. His calculation ended with the following inequality:

$$314 \frac{63}{625} = A_{192} < A < A_{96} + 2(A_{192} - A_{96}) = 314 \frac{169}{625}$$

Unlike Archimedes, Liu Hui obtained his upper limit from the inscribed polygon with double the number of sides. At the end of his calculation, Liu Hui established the following approximation from his two limits, which differ by $\frac{105}{625}$ and then arrived at the following approximation for π which differs from the true value by $7.346 \cdot 10^{-6}$.

$$\pi \approx \frac{314 + \frac{4}{25}}{10^2} = 3.1416$$

The second important ancient Chinese π scholar was Tsu Chhung-Chih or Zu Chongzhi (429-500). He improved the precision of π from Liu's value by around 2 powers of ten to the limits $3.1415926 < \pi < 3.1415927$. This interval is accurate to 7

decimal places and it held the world record for 800 years. Besides that, Tsu Chhung-Chih was one of the first to express these numbers using decimal notation (Volkov, 1997). He wrote:

3 zhang, 1 chi, 4 cun, 1 fen, 5 li, 9 hao, 2 miao, 7 hu

The expression above, zhang, chi, etc. are all units of length which behave like 1:10:100:..... Tsu also found a second value which was numerically less accurate but more visually attractive that is :

$$\pi \approx \frac{355}{113} = 3.1415929.....$$

This decimal fraction, which is accurate to 6 decimal places, agrees with the fourth partial quotient of the simple continued fraction of π .

c. India

In 499 A.D., the astronomer Aryabhata wrote a work called *Aryabhatiya*. His theorem 10 out of 133 theorems goes as follows (Katz, 1993):

“Add 4 to 100, multiply by 8 and add 62,000. This is the approximate circumference of the circle of which the diameter is 20,000.”

This produces a value of π of $\frac{62832}{20000} = 3.1416$. However, this value is suspected of

being of Greek origin and having found its way to India like many other things. But Aryabhata may have calculate the value himself, for he demonstrated that if a is the length of the polygon with n sides which is inscribed in a circle of diameter 1, and b is the length of a side of a polygon with twice as many sides $2n$, then

$b^2 = \left(1 - \sqrt{1 - a^2}\right) / 2$. He begun his calculation with a hexagon then followed by

polygons with 12,24,....,384 sides. The perimeter of the last polygon which is $\sqrt{9.8694}$ is where he calculated the approximate value of $\pi = 3.1416$ (Rouse Ball, 1974).

The Hindu Brahmagupta was interested by the discovery that the perimeter of regular polygons with 12, 24, 48 and 96 sides and diameter 10 are $\sqrt{965}$, $\sqrt{981}$, $\sqrt{986}$ and $\sqrt{987}$. From this, he concluded that if the number of sides were doubled again, the perimeter would tend towards $\sqrt{1000}$. Using this logic, he arrived at $\pi = \sqrt{1000}/10 = \sqrt{10} = 3.16227...$ (Castellanos, 1988).

d. Choresmia (today Uzbekistan)

In c. 830, the Choresmian Alkarism made his mark on the history of π by coming up with the three values of π which is $\frac{22}{7}$, $\sqrt{10}$ and $\frac{62832}{20000}$. The first value was intended to serve as a mean value, the second was meant for geometricians and the third for astronomers.

e. Europe

In around 1220, Leonardo of Pisa or better known as Fibonacci, independently calculated the value of $\pi = \frac{864}{275} = 3.14181...$ from a 96-sided polygon, without drawing on Archimedes's work. His approach was more accurate than Archimedes but

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