CATALAN NUMBERS

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A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF BACHELOR OF SCIENCE WITH HONOURS

MATHEMATICS WITH ECONOMICS PROGRAM
SCHOOL OF SCIENCE AND TECHNOLOGY
UNIVERSITI MALAYSIA SABAH

2007

BORANG PENGESAHAN STATUS TESIS@

JUDUL: CATALAN NUMBERS

Ijazah: SARJANA MUDA SAINS DENGAN KEPUJIAN (MATEMATIK Dengan EKONOMI)

SESI PENGAJIAN: 2004/2005

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ACKNOWLEDGEMENTS

I would first and foremost like to thank my supervisor, Mr. Rajasegeran for giving me the guidance and motivation needed to complete this thesis and for his efforts in ensuring that all his students get the utmost from his knowledge and experience.

I would also like to thank my family and friends, who inspired and helped me throughout finishing this thesis.

Last but not least, I thank God for giving me continual wisdom, guidance and support in my life as well as studies.

ABSTRACT

This dissertation is about Catalan number that can form a sequence of natural numbers. The nth Catalan number is given in the terms of binomial coefficients :

$C_n = \frac{1}{n+1}\binom{2n}{n}$. There are two methods that are used to derive the nth Catalan number are elaborated. The first method is using recurrence relation and the second method is by using the bijective proof. The relation between Catalan number and group theory is explained. The application of Catalan number in group theory is shown. Catalan number is applied in group theory using balanced parentheses. Catalan number is used to see the different ways of groupings. By using Catalan numbers, the numbers can be multiplied in many orders without changing the orders of the numbers. The grouping using Catalan numbers can only be applied to addition and multiplication. In this dissertation also, various Catalan number problems have been researched. This includes nonintersecting chords problem, polygon triangulation, mountain ranges and diagonal avoiding path. The relation between Catalan numbers and group theory is shown by solving these problems.

NOMBOR CATALAN

ABSTRAK

Dissertasi ini adalah mengenai nombor Catalan yang membentuk siri nombor asli. Nombor Catalan yang ke n diberi menggunakan formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. Dua kaedah yang boleh digunakan untuk memperoleh nombor Catalan diterangkan secara mendalam. Kaedah yang pertama adalah melalui "recurrence relation" dan kaedah kedua ialah dengan menggunakan pembuktian bijektif. Hubungan antara nombor Catalan dan "group theory" telah diperolehi. Aplikasi nombor Catalan dalam "group theory" diperjelaskan secara mendalam. Ianya diaplikasikan dengan menggunakan "balanced parentheses". Nombor Catalan digunakan untuk memperoleh pelbagai susunan kumpulan tanpa memjejaskan urutan. Ia hanya dapat diaplikasikan untuk penambahan dan pendaraban. Dissertasi ini turut memperlihatkan pelbagai masalah nombor Catalan. Ini termasuk "nonintersecting chords problem", "polygon triangulation", "mountain ranges" dan "diagonal avoiding path". Hubungan antara nombor Catalan dan "group theory" dapat dilihat melalui penyelesaian masalah nombor Catalan.

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LIST OF SYMBOLS

$=$	Equal sign
$\rightarrow$	Arrow
$\geq$	Greater than or equal to
$+$	Addition sign
$-$	Subtraction sign
$/$	Division sign
$\sqrt{\ }$	Square root sign

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

The Catalan sequence was first described in the 18^{th} century by Leonhard Euler, trying to solve the problem of subdividing polygons into triangles. The sequence is named after Èugene Charles Catalan who discovered the connection to parenthesized expressions.

Catalan numbers are integer sequence $\{C_n\}$ which appears in tree enumeration which is the Euler's polygon division problem. According to combinatorial mathematics, Catalan numbers form a sequence of natural numbers and often involve recursively defined objects. The *n*th Catalan number is given in the terms of binomial coefficients:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \quad \text{for} \quad n \geq 0 \qquad (1.1)$$

The first few numbers are 1, 2, 5, and 14. C_2 and C_3 are Catalan primes so therefore C_3 which is equals to 5 is the largest Catalan primes. The odd Catalan number are in the form of C_{2^k-1}. The first few are 1, 5, 429, 9694845, ... Interestingly the last digit of odd Catalan numbers are 1, 5, 9, 5, 9, 5, 9, 7, 5, 5, 5, 5, 5, ... and if we observe carefully 5 is the last digit from $k = 9$ up to at least $k = 30$.

The generating function for the Catalan numbers is given as:

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{x=0}^{\infty} C_n x^n = 1 + x + 2x^2 + 5x^3 + ... \tag{1.2}$$

Catalan numbers satisfy the recurrence relation which can be shown as below:

$$C_0 = 1$$

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ for } n \geq 0 \tag{1.3}$$

Asymptotically, the Catalan numbers grow as the quotient of the n^{th} Catalan number and the expression on the right tends towards 1 for $n \to \infty$.

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}} \tag{1.4}$$

Catalan numbers can be interpreted in various ways. There are at least about 66 different ways to interpret Catalan numbers. The interpretation for nth Catalan numbers includes the number of ways to arrange n pairs of matching parentheses in which it can be placed in a sequence of numbers to be multiplied by two at a time and also the number of ways a convex polygon of $n+2$ sides can be split into n triangles. Other than that the interpretation also includes the number of planar binary trees with exactly $n+1$ leaves and the number of paths of length $2n$ through an $n\times n$ grid that do not rise above the main diagonal (Wikipedia, 2006).

Catalan number problems also include the Catalan number, C_{n-1} which gives the number of bracketings of n letters and also the solution to the ballot problem. Catalan numbers can be used to solve the number of mountains which can be drawn with n upstrokes and n down strokes and the number of non-crossing handshakes possible across a round table between n pairs of people and the number of sequences with nonnegative partial sums which can be formed from n 1s and n-1s (Wikipedia,2006).

1.3 RESEARCH OBJECTIVE

The research has few objectives:

i. To study and understand Catalan numbers.
ii. Using two different ways to deduce Catalan Numbers.
iii. To find the applications of Catalan numbers in the field of mathematics.
iv. To find the applications of Catalan numbers in group theory.

1.4 RESEARCH SCOPE

The research scope focuses mainly on finding the solution using Catalan numbers for combinatorics counting problems and its interpretations. Other than that, the scope of the research also includes the applications of Catalan numbers, hence having a better understanding of Catalan numbers.

CHAPTER 2

LITERATURE REVIEW

2.1 HISTORY

Èugene Charles Catalan was born in Brugge, Belgium. He is the only child of French jeweler by the name of Joseph Catalan. Catalan was born in 1814. He traveled to Paris in 1825 and learned mathematics at École Polytechnique. In 1834 he was expelled from university but in 1841 with the help of Liouville, he obtained his degree in mathematics. He taught descriptive mathematics in college. He was active in politics and participated in 1848 revolution. After the revolution he was elected as the member of the France's Chamber of Deputies. The University of Liège appointed him chair of analysis in 1865. He then became a journal editor before working for the Belgian Academy of Science in the field of number theory. He died in Liège, Belgium (Wikipedia, 2006b).

Catalan worked on continued fractions, descriptive geometry, number theory and combinatorics. He gave his name to a unique surface that he discovered in 1855. Before that, he had stated the famous Catalan's conjecture, which was published in 1844 and was eventually proved in 2002, by the Romanian mathematician

Preda Mihăilescu. The conjecture shows that 8 and 9 which is 2^3 and 3^2 are the only consecutive powers, excluding 0 and 1. This conjecture is the only nontrivial solution to Catalan's Diophantine problem. He introduced the Catalan numbers to solve a combinatorial problem (Wikipedia, 2006).

Photo 1.1 Èugene Charles Catalan

2.2 INTRODUCTION

Catalan numbers can be considered a new addition to number theory compared to other number sequences such as Fibonacci numbers and Ulam numbers. Catalan number has only been around since 1814. Through out the years many works and research had been done using Catalan numbers which includes the use of Catalan numbers in conjectured statistics, Hankel matrix and continued fraction.

2.3 CONJECTURED STATISTICS

A rational function $C_n(q,t)$ which conjectured always evaluates to a polynomial in q and t with nonnegative coefficients was introduced. It was proved that $C_n(q,t)$ is always a polynomial but with possibly negative coefficients. An elementary proof of this result has been given but the nonnegativity remains open. Other conjectures have $C_n(q,t)$ related to the Frobenius series of a bigraded S_n module. By modifying a proof of a q-Langrange inversion formula based on results, a q-analogue of the general Langrange inversion formula which involves Catalan numbers (Haglund, 2002).

2.4 HANKEL MATRIX

Hankel matrix is named after Hermann Hankel. A square matrix with constant skew diagonals is a Hankel matrix. As for example :

$$\begin{bmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{bmatrix}$$

Otherwise in mathematical terms:

$$a_{ij} = a_{i-1\,j+1} \tag{2.1}$$

In other words, a Hankel matrix is a matrix in which the (i,j)th entry depends on the sum of $i+j$. Such matrices are sometimes known as persymmetric matrices. Hankel matrix is closely related to the Toeplitz matrix. A Hankel matrix is an upside-down of Toeplitz matrix. The elements of Hankel matrix is given as:

$$h_{ij} = \begin{cases} 0 & ,i+j-1>n \\ i+j-1 & ,otherwise \end{cases} \tag{2.2}$$

The $n \times n$ Hankel matrix whose (i,j) entry is the Catalan number C_{i+j} has determinant 1, regardless of the value of n. As an example, for n=3, the determinant:

$$\det\begin{bmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{bmatrix} = 1$$

Therefore Catalan number is a unique sequence with this property (Wikipedia, 2006a).

2.5 DETERMINANT PROPERTY OF CATALAN NUMBERS

Catalan numbers arise in a family of persymmetric arrays with determinant 1. The demonstration involves a counting result for disjoint path systems in acyclic directed graphs. The Catalan number C_n is the number of all sequences such that C_0 is the number of empty sequences so $C_0 = 1$. A persymmetric matrix is a square matrix with constant skew diagonals. Such matrices were also called orthosymmetric (Mays & Jery, 1999).

2.6 CATALAN-LIKE NUMBERS AND DETERMINANTS

A class of numbers, called Catalan-like numbers are introduced which unify many well known counting coefficients, such as Catalan numbers, the Motzkin numbers, the middle binomial coefficients, the hexagonal numbers and many more. Generating functions, recursions and determinants of Hankel matrices are computed and some other interpretations are given as to what these numbers count. There are several formulae for the Catalan numbers involving binomial coefficients. Catalan numbers, C_n and Motzkin numbers, M_n have several common properties. The numbers C_n are the unique sequences of real numbers such that the Hankel matrices each have determinant 1 for all n. It is also proven that for the Motzkin numbers, the determinant of the first Hankel matrix is again 1 for all n (Aigner, 1998).

2.7 CATALAN CONTINUED FRACTION

Continued fraction with monic monomial numerators is a Catalan continued fraction. Let $\varepsilon_k(\pi)$ be the number of increasing subsequences of length k+1 in the permutation π. It is proven that any Catalan continued fraction is multivariate generating function of a family of statistics on the 132-avoiding permutations where each consists of a linear combination. Moreover, there is an invertible linear transformation that translates between linear combinations and the corresponding continued fractions (Jani & Rieper, 2002).

2.8 CONTINUED FRACTIONS AND CATALAN PROBLEMS

A Catalan problem is any enumerative problem that produces the Catalan sequence of numbers or one of its many q-analogs. Interestingly, many of the generating functions that arise from these problems can be given as a continued fraction with a simple yet elegant form. A generating function expressed as a continued fraction that enumerates ordered trees by the number of vertices at different levels. Several Catalan problems are mapped to an ordered-tree problem and their generating functions also expressed as a continued fraction. Among these problems is the enumeration of avoiding permutations that have a given number of increasing patterns of certain length (Rieper, 2000).

2.9 FACTORIZATIONS

Another occurrence of the Catalan numbers is shown by factorizations. The number of primitive factorizations of the cyclic permutation (1, 2, ..., n+1) into n transpositions is C_n, the n-th Catalan number. A factorization $((a_1 \quad b_1),(a_2 \quad b_2),...,(a_n \quad b_n))$ is primitive if its transpositions are ordered, in the sense that the $a_i s$ are non-decreasing. It is proven that the sequence counting primitive factorizations satisfies the recurrence for Catalan numbers and an explicit bijection between the set of primitive factorizations and the set of avoiding permutations, known to have size counted by Catalan numbers (Merola, 2005).

2.10 AREA OF CATALAN PATH

Among many other combinatorial structures, the nth Catalan number enumerates the number of lattice paths of length n, in the plane $Z \times Z$ from (0,0) to (n,n) using north steps (0,1) and east steps (1,0) that never pass below the line y = x. Let C_n denote the set of Catalan paths of length n. A Catalan path is said to be elevated if it remains strictly above the line y = x except at the start and end of points. The area of a Catalan path is defined to be the number of triangles of the region enclosed by the path y = x.

It is known that the area of all Catalan paths of length n is equal to $4^n - \binom{2n-1}{n}$ which coincides with the number of inversions of all avoiding permutations of length $n+1$. In addition a bijection between the two sets is established (Cheng & Eu, 2006).

2.11 TRIANGULATIONS

There are many combinatorial objects counted by the Catalan numbers. Some of them have been widely studied, especially binary trees and lattice paths. One of the latter is the case for the triangulations of a convex polygon, it's a problem that goes back to Euler. It is the number of ways of triangulating a convex polygon of n sides by means of non-intersecting diagonals. It is known that a convex polygon of n sides admits C_{n-2} triangulations, where C_n is a Catalan number (Noy & Hurtado, 1994). These triangulations can be classified according to their dual trees and prove the following formula for the number of triangulations of a convex polygon whose dual tree has exactly k leaves:

$$\frac{n}{k}2^{n-2k}\binom{n-4}{2k-4}C_{k-2} \tag{2.3}$$

The proof is bijective and provides a recursive formula for the Catalan numbers similar to, but different from a classical identity of Touchard, An averaging argument allows to deduce Touchard's formula (Hurtado , 1994).

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