ACCELERATED OVERRELAXATION (AOR) METHOD IN SOLVING FUZZY SYSTEM OF LINEAR EQUATIONS

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THIS DISSERTATION IS SUBMITTED IN PARTIAL FULFILLMENTS OF THE REQUIREMENTS FOR THE BACHELOR OF SCIENCE DEGREE WITH HONOURS

MATHEMATICS WITH COMPUTER GRAPHICS PROGRAMME
SCHOOL OF SCIENCE AND TECHNOLOGY
UNIVERSITI MALAYSIA SABAH

APRIL 2007

PUMS 99:1

UNIVERSITI MALAYSIA SABAH

BORANG PENGESAHAN STATUS TESIS@

JUDUL: ACCELERATED OVERRELAXATION METHOD (AOR) IN SOLVING FUZZY SYSTEM OF LINEAR EQUATIONS

Ijazah: SARJANA MUDA SAINS DENGAN KEPUJIAN (MATEMATIK DENGAN KOMPUTER GRAFIK)

SESI PENGAJIAN: 2004-2007

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DECLARATION

I declare that this writing is my own work except for citations and summaries where the sources of every each have been duly acknowledged.

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ACKNOWLEDGEMENT

Firstly, I would like take this golden opportunity to thank my supervisor, Assoc. Prof. Dr. Jumat Sulaiman for his kind help and guidance throughout the process of completing this dissertation. He has shared his knowledge and ideas which have guide me well to detect and solve the errors and doubt that occurs.

I would also like to take this chance to show my appreciation to all the Mathematics lecturers whom have taught me throughout these 3 years. Their guidance and knowledge had been the very first step for me in exposure towards the field of mathematics in higher level and industrial application.

Lastly, thanks to my family members and friends who have supported me financially and morally by giving confidence and advice in order to complete this dissertation. Thus, I shall shower my gratitude to those who have supported along the way of completing this dissertation.

ABSTRACT

In most of the science and engineering field, the linear system plays a very essential role, mainly in the applications sectors. However, several of the applications use fuzzy numbers rather than the crisp number for the precision reason. Thus, it is very important to develop numerical procedures that can appropriately solve general fuzzy linear systems. This study was carried out with the objective to propose the Accelerated Overrelaxation (AOR) algorithm for solving mainly the Fuzzy System of Linear Equations (FSLE). FSLE was derived into matrix notation using the embedding method. Embedding method facilitates to extend the fuzzy linear system into crisp linear system where the generated matrix size will be double the size of the original fuzzy linear system. In this study, only square matrices were taken into consideration. The extended matrices were solved using the control method, Gauss-Seidel (GS) and the proposed method, AOR method. By substituting appropriate values for all the parameters involved such as the overrelaxation parameter and acceleration parameter, the optimal solutions were obtained for both control and proposed methods. These two methods were also compared in the aspects of number of iterations and execution time. Based on the results, it was concluded that AOR method converges faster with lesser number of iterations and execution time than the GS method with a better convergence rate.

KAEDAH PENGENDURAN BERLEBIHAN PECUTAN (PBP) DALAM MENYELESAIKAN SISTEM PERSAMAAN LINEAR KABUR

ABSTRAK

Sistem linear memainkan peranan penting dalam kebanyakan aplikasi, terutamanya dalam bidang sains dan kejuruteraaan. Akan tetapi, dalam kebanyakan aplikasi nombor kabur lebih gemar digunakan berbanding dengan nombor asli. Oleh itu, ia adalah penting untuk menyelesaikan sistem linear kabur dengan menggunakan kaedah berangka. Kajian ini telah dijalankan dengan objektif untuk mengemukakan kaedah lelaran, Pengenduran Berlebihan Pecutan (PBP) untuk menyelesaikan Sistem Persamaan Linear Kabur (SPLK). SPLK akan dideduksikan kepada bentuk matriks dengan menggunakan kaedah benaman. Kaedah ini memudahkan untuk membentuk SPLK kepada sistem persamaan linear biasa dengan saiz matriks yang telah dikembangkan adalah dua kali ganda dari saiz matriks asal. Sebenarnya, dalam kajian ini, hanya matriks segiempat sama dipertimbangkan dan diselesaikan dengan menggunakan kaedah kawalan, Gauss-Seidel (GS) dan kaedah PBP. Penyelesaian optimum dapat diperolehi dengan penggantian nilai yang sesuai untuk parameter-parameter yang terlibat. Kedua-dua kaedah ini kemudian dibandingkan dari segi bilangan lelaran dan masa lelaran. Berdasarkan keputusan, dapat disimpulkan bahawa kaedah PBP boleh menumpu dengan lebih cepat dengan bilangan lelaran dan masa lelaran yang lebih kecil berbanding dengan kaedah GS. Malah, PBP mempunyai kadar penumpuan yang lebih baik daripada GS.

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$+$	Addition
$-$	Subtraction
$\times$	Multiplication
$=$	Equal
$\neq$	Not equal
$>$	Greater than
$\geq$	Greater or equal
$<$	Less than
$\leq$	Less or equal
$\rightarrow$	Implication
$\in$	Element
$\notin$	Not an element
$\oplus$	Exclusive
$\otimes$	Inclusive
ε	Error
C	Complex number
$\widetilde{x}$	Fuzzy
$\sum$	Summation
$x^{(n)}$	The n^{th} iteration of x

CHAPTER 1

INTRODUCTION

1.1 Artificial Intelligence

According to Badiru and Cheung (2002), natural intelligence involves the capability of humans to acquire knowledge, reason with the knowledge and use it to solve problems effectively. By contrast, artificial intelligence (AI) is defined as the ability of a machine to use simulated knowledge in solving problems. Hence, AI is used to provide tools for reasoning and also to understand complex or incomplete phenomena (Schneider *et al.*, 1996).

1.1.1 Origin of Artificial Intelligence

Actually, the definition of intelligence had been sought by many great philosophers and mathematicians over the ages, including Aristotle, Plato, Copernicus and Galileo. They tried to explain the process of thought and understanding. But, Thomas Hobbes started the revolution for the simulation of intelligence by putting forward an interesting concept in the 1650s. Hobbes believed that thinking consists of symbolic operations and everything in life can be represented mathematically. This directly led to the idea that a machine is capable of carrying out mathematical operations on symbols that could imitate human thinking. This idea became the basic driving force behind AI effort (Badiru & Cheung, 2002).

The first attempt to establish the field of machine intelligence into an organized effort occurred during 1956 summer with the organization of The Dartmouth Summer Conference by John McCarthy, Marvin Minsky, Nathaniel Rochester and Claude Shannon. This conference has brought together people whose work and interest formally founded in the field of AI and it was this conference also, where John McCarthy coined the term "artificial intelligence" (Leondes, 2002).

1.1.2 Branches of Artificial Intelligence

Current subspecialties of AI include (Badiru & Cheung, 2002):

i) *Natural language processing* - deals with various areas of research such as database inquiry systems, story understanders, automatic text indexing, grammar and style analysis of text, automatic text generation, machine translation, speech analysis and speech synthesis.

ii) *Computer vision* - deals with research efforts involving scene analysis, image understanding and motion derivation.

iii) *Robotics* - involves the control of effectors involving on robots to manipulate or grasp objects, locomotion of independent machines and use of sensory input to guide actions.

iv) *Problem-solving and planning* - involves applications such as refinement of high-level goals into lower-level ones, determination of actions needed to achieve goals, revision of plans based on intermediate results and focused search of important goals.

v) *Learning* - deals with research into various forms of learning including rote learning, learning through advice, learning by example, learning by task performance and learning by following concepts.

vi) *Expert systems* - deal with the processing of knowledge as opposed to the processing of data. It involves the development of computer software to solve complex decision problems.

The most significant success has been developed of the very powerful AI tools is expert systems (ES). This ES incorporates the reasoning of human experts in a particular domain and combines it with the computer's processing speed and large memory capacity. ES is constructed to mimic the reasoning process of human experts, to capture and preserve their knowledge and to make this knowledge more accessible. ES are sophisticated programs that utilize powerful computer resources to assist human experts in performing their tasks. Unlike human experts, ES do not learn from experience, they are incapable of making intelligent guesses or applying common sense and they cannot use rule of thumb in their decision-making procedures (Schneider *et al.*, 1996). When some uncertainty occurs in expert system, it requires the techniques of fuzzy system similar to the other branches of AI.

1.2 From Classical Set Theory to Fuzzy Set Theory

Classical set theory establishes systematic relations among objects within a set as well as between elements of various sets. A set is a collection of any number of definite, well distinguished objects, called the elements of the sets. Thus, an object either belongs to the set or be completely excluded from it (Badiru & Cheung, 2002).

Methods based on classical set theory are utilized mainly in areas where measurements can be made very precisely. However, when such favourable conditions are not reflected in the domain of the problem, the application of classical set theory does not yield good result. Thus, the use of classical set theory to classify data and knowledge may lead into distortion during interpretation of data or knowledge. It can be even severe in cases where information available for the decision making process is difficult to obtain in precise mathematical form. Therefore, decision must be made based on data containing some degree of error (Schneider *et al.*, 1996).

Due to the above phenomena, it has led to the development of fuzzy set theory, first introduced by Lotfi A. Zadeh. Fuzzy set theory differs from classical set theory from one main aspect: An element can either belong to the fuzzy set, be completely excluded from the fuzzy set or belong to the fuzzy set to any intermediate degree between these two extremes. The extent to which an element belongs to a given fuzzy set is called the grade of membership or degree of membership (Schneider *et al.*, 1996).

The term fuzzy was introduced to describe sets whose membership criteria are vague. For instance, warm water is only marginally a member of the set of cold water. Uncertainty about a statement such as the water is cold is not represented by the probability but possibility in fuzzy set. The possibility of a statement is represented by a number generated by a membership function (Schneider *et al.*, 1996). Thus, membership functions of a set A in a classical set is

$$\mu_A(x)=\begin{cases}1 & if \;\; x\in A \\ 0 & if \;\; x\notin A\end{cases}.$$

In a fuzzy set, the membership value, $\mu_A(x)$ takes any value in the interval $[0,1]$.

1.3 Fuzzy Systems

Since many real-world engineering systems are too complex to be defined in precise terms, imprecision is often involved in any engineering design process. Fuzzy systems have an essential role in this modeling, which can formulate uncertainty in actual environment (Dehghan *et al.*, 2006). Fuzzy systems also provide an alternative approach in representing problems and processing information. While there are many computational algorithms developed to process numerical data, fuzzy systems provide an alternative way to manipulate information, not just data only (Badiru & Cheung, 2002).

In 1965, fuzzy logic was first proposed by Lotfi A. Zadeh of the University of California at Berkeley. He elaborated on his ideas in a 1973 paper that introduced the concept of "linguistic variables", which associate to a variable defined as a fuzzy set. Since then, fuzzy logic has emerged as a powerful technique for the controlling industrial processes, household and entertainment electronics, diagnosis systems and other expert systems. The rapid growth of this technology has actually started from Japan and then spread to the America and Europe (Schneider *et al.*, 1996).

Interest in fuzzy systems was sparked by Seiji Yasunobu and Soji Miyamoto of Hitachi, who provided simulations that, demonstrated the superiority of fuzzy control systems for the Sendai railway. Their ideas were adopted and fuzzy systems were used to control accelerating, braking, and stopping in 1987 (Schneider *et al.*, 1996).

In order to construct a fuzzy system, it is essential to obtain a collection of IF-THEN rules from human experts or based on domain knowledge and later combines these rules into a single system. Three common types of fuzzy system in literature are pure fuzzy systems, Takagi-Sugeno-Kang (TSK) fuzzy systems and fuzzy systems with fuzzifier and defuzzifier (Schneider *et al.*, 1996). The algebras of a fuzzy system are shown in Figure 1.1;

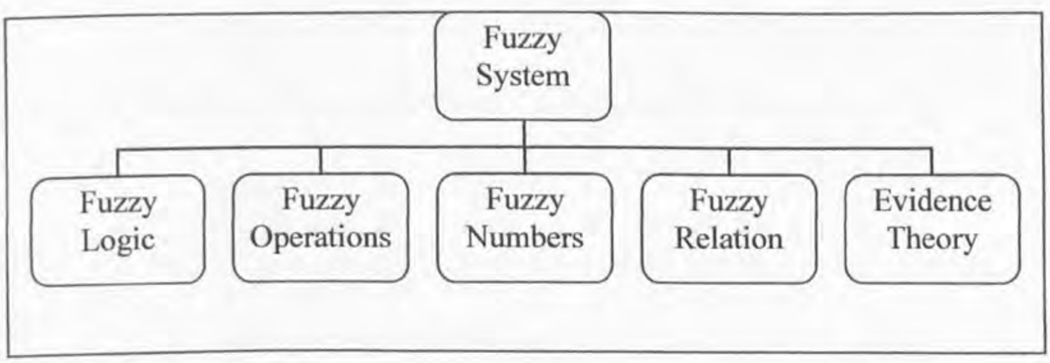

Figure 1.1 Algebra of a fuzzy system

1.3.1 Fuzzy Logic

Fuzzy logic is a technique for dealing with sources of imprecision and uncertainty that are non statistical in nature. It uses a multi valued membership function to denote membership of an object in a set. Fuzzy logic also aimed at providing a body of concepts and techniques for dealing with approximate modes of reasoning rather than exact ones (Badiru & Cheung, 2002).

In fuzzy logic, an ambiguous term fuzzy set is represented which have been discussed in Section 1.4. Extensions of fuzzy sets now include concepts such as fuzzy arithmetic, possibility distributions, fuzzy statistics, fuzzy random variables and fuzzy set function (Leondes, 2002).

1.3.2 Fuzzy Operations

Fuzzy operations consist of (Badiru & Cheung, 2002):

i) Fuzzy complement – if the original fuzzy set points to the degree of participation in the universe of discourse, the complement points to the degree of nonparticipation in the universe of discourse.

   $c:[0,1]\rightarrow[0,1]$

   Examples of fuzzy complements are Threshold complement, Standard complement, Sugeno Class complement and Yager class complement.

ii) Fuzzy intersection – the intersection of two fuzzy sets in a third fuzzy set that contains the elements common to two original sets.

   $i:[0,1]\times[0,1]\rightarrow[0,1]$

The defining conditions for a fuzzy intersection, also called as *t*-norm or triangular norm, are the boundary conditions obtained from crisp logic with the combination of properties of monotonicity, commutativity and associativity. Examples of fuzzy intersection are Drastic, Algebraic product, Bounded difference and Yager intersections.

iii) Fuzzy union – the union of two fuzzy sets in a third fuzzy set that contains the elements that may be included in either of the two original sets.

$$u:[0,1]\times[0,1]\rightarrow[0,1]$$

The defining conditions for a fuzzy union, also called as *t*-conorm, are the boundary conditions obtained from crisp logic with the combination of properties of monotonicity, commutativity and associativity. Examples of fuzzy unions are Drastic, Algebraic sum, Bounded sum, Sugeno class and Yager class unions.

iv) Duality – a duality property can be expressed in terms of De Morgan's Law, similar to crisp logic which is extended using fuzzy operators.

v) Fuzzy implication – the implication of two fuzzy sets results in a third fuzzy set that conveys the meaning of the first set implying for the second set.

$$u:[0,1]\times[0,1]\rightarrow[0,1]$$

The defining conditions for a fuzzy implication are the boundary conditions obtained from crisp logic with the combination of property of monotonicity. Examples of fuzzy implications are Godel, Lukasiewicz, Zadeh, Kleene-Dienes implications.

vi) Fuzzy aggregation – an operation that produces a value in the middle range between the intersection and union operations.

$h : [0,1]^n \rightarrow [0,1]$

Examples of aggregation operations are Generalized, Harmonic, Arithmetic and Ordered weighted average means.

1.3.3 Fuzzy Numbers

Fuzzy numbers are an application of fuzzy concepts. An arbitrary fuzzy number in parametric form is a pair $(\underline{u}, \overline{u})$ of function $\underline{u}(r), \overline{u}(r), 0 \leq r \leq 1$, which satisfies the following requirements (Zheng & Wang, 2006a):

i) $\underline{u}(r)$ is a bounded left non-decreasing function over $[0,1]$.

ii) $\overline{u}(r)$ is a bounded left non-increasing function over $[0,1]$.

iii) $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1$.

Simple arithmetic can be done on fuzzy numbers such as addition, subtraction, multiplication and division. As for other operations, it is possible with the concept of α-cut and extension principle (Schneider *et al.*, 1996).

1.3.4 Fuzzy Relation

In fuzzy relations, the relation can be defined by the membership matrix method with the elements value between 0 and 1, as for crisp relation the value is either 0 or 1. Even for crisp relation, the existence of line indicates whether a relation exist or not but for fuzzy relation, the degree of strength of the relation is indicated in line. This relation is used to find the value of a resultant fuzzy set given the value of an initial fuzzy set called the max-min composition method (Badiru & Cheung, 2002).

1.3.5 Evidence Theory

In evidence theory, a belief measure is the degree of belief based on available evidence that a given element belongs to the set A. The basic principle of the belief measure is that the sum is greater than the parts. Probability is the special case of belief measure. While the belief measure deals with hard evidence, the plausibility measure deals with what can be implied or inferred from the evidence. The belief measure is superadditive whereas plausibility measure is subadditive. Similar to intersection and union, duality does exist between belief and plausibility measure also (Badiru & Cheung, 2002).

The nonspecificity or imprecision of a fuzzy set can be measured in terms size of alternatives. Another uncertainty measure is the fuzziness measure which is a measure of a lack of distinction between the set and its complement. The less a set differs from its complement, the fuzzier it is. As comparison, a gain in information always reduces the nonspecificity, but a reduction in fuzziness does not necessarily imply a gain in information (Badiru & Cheung, 2002).

1.4 Fuzzy System of Linear Equations (FSLE)

Systems of simulations linear equations play major role in various areas such as mathematics, physics, statistics, engineering and social sciences. Since in many applications at least some of the system's parameters and measurements are represented by fuzzy rather than crisp numbers, it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve those (Abbasbandy *et al.*, 2006).

One of the main applications using fuzzy number arithmetic is treating linear systems whose parameters are all partially represented by fuzzy number. The $m \times n$ linear system

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = y_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = y_m \end{array}\right\} \tag{1.1}$$

where the coefficients $A = (a_{ij})$ is a crisp (discrete) numbers and y_i, $i = 1, 2, ..., m$ are fuzzy numbers is called a fuzzy linear system (Friedman *et al.*, 1998).

1.5 Fully Fuzzy Linear Systems

Another type of fuzzy linear systems is fully fuzzy linear system (FFLS), in the case that all the parameters involved are fuzzy numbers, compared to FSLE only the right-hand-side. The $n \times n$ linear system of equations:

$$\left.\begin{array}{l} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1 \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2 \\ \vdots \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n \end{array}\right\} \tag{1.2}$$

where the coefficients $\tilde{A} = (\tilde{a}_{ij})$, $1 \le i$, $j \le n$ is a n x n fuzzy matrix, $\tilde{x}_i$ and $\tilde{b}_i$ are elements of the class of a fuzzy set, is called a fully fuzzy linear system (Dehghan *et al.*, 2006).

1.6 Numerical Methods for Solving FSLE

Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations where they invariably involve large numbers of tedious arithmetic calculations. They also capable of handling large systems of linear systems of equations, nonlinearities and complicated geometries that are common in engineering practice and that are often impossible to solve analytically. In order to solve systems of linear equation,

$$Ax = b \qquad (1.3)$$

there are two types of solution methods which are called the direct method and iterative method (Chapra & Canale, 1985).

1.6.1 Direct Method

This method normally focus in getting the inverse of a coefficient matrix of A and no initial value required during calculation (Gerald & Wheatly, 1999). The variations of the direct method, together with its examples will be illustrated in Figure 1.2 (Saad, 1996).

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