THE HANOI GRAPHS FOR THE HORIZON TOWER

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DECLARATION

I hereby declare that this dissertation contains my original research work. Sources of findings reviewed herein have been duly acknowledged.

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ABSTRACT

This dissertation describes a variant of the Tower of Hanoi game called The Horizon Tower where this game contains different disks arrangements. The disks in the Horizon Tower are given two alternating colours (blue and red) and hence this game can be solved to build one Hanoi Tower or two Hanoi Towers according to the disk colour. The effects of the different disk arrangements were analyzed using the Hanoi graphs for the Horizon Tower. The analysis shows that different disk arrangement of the Horizon Tower alters the shape of the Hanoi graph. The Hanoi graphs for the Horizon Tower also varies as the number of disks increases. Hence, a comparison between the graphs was made and the results show that there are some similarities between the graphs. However, as the number of disks increases, the dissimilarity between the graphs increases. The graphs become more complicated and more difficult to create. The Hanoi graph for the Horizon Tower was also used in evaluating the fastest way in obtaining the solutions to the problem. This research shows that the Horizon Tower is easier solved as the number of moves required to solve this game is much lesser compared to the Tower of Hanoi.

GRAF HANOI UNTUK MENARA UFUK

ABSTRAK

Disertasi ini memperihalkan satu variasi permainan Menara Hanoi yang diberi nama Menara Ufuk di mana permainan ini mempunyai susunan cakera yang berlainan. Cakera-cakera permainan Menara Ufuk ini diberi dua warna bergilir (biru dan merah), oleh itu permainan ini boleh diselesaikan untuk membentuk satu Menara Hanoi atau dua Menara Hanoi mengikut warna. Kesan susunan cakera yang berlainan ini dikaji menggunakan graf Hanoi bagi Menara Ufuk. Analisis ini menunjukkan bahawa susunan cakera yang berlainan bagi Menara Ufuk mengubah bentuk graf Hanoi. Bentuk graf Hanoi ini juga berubah apabila bilangan cakera ditingkatkan. Maka, satu perbandingan di antara graf-graf yang terhasil dibuat. Hasil ini menunjukkan bahawa terdapatnya beberapa persamaan di antara graf-graf tersebut. Namun, semakin banyak bilangan cakera ditambah, semakin banyak perbezaan antara graf-graf tersebut wujud. Grafnya juga menjadi lebih rumit dan sukar untuk dihasilkan. Graf Hanoi untuk Menara Ufuk ini juga telah digunakan dalam mengkaji penyelesaian yang terpantas untuk permainan ini. Kajian ini menunjukkan bahawa Menara Ufuk dapat diselesaikan dengan mudah kerana bilangan pergerakan cakera yang diperlukan untuk menyelesaikan permainan ini adalah jauh lebih rendah apabila dibandingkan dengan Menara Hanoi.

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${}^{n}C_{r}$	Combination of choosing r objects from n objects

CHAPTER 1

INTRODUCTION

1.1 Introduction

There are many interesting mathematical games or puzzles around the world and among these remarkable games is a game called The Tower of Hanoi. This game consists of disks of different diameter sizes which are arranged on a pole or peg in a manner that, once all disks are arranged, it will combine to look like a tower. There are a total of three pegs in the Tower of Hanoi game where at the beginning of the game the disks are arranged on one peg which is then moved until the all disks are arranged on another peg at the end of the game.

Invented by a French mathematician, this game caught not only the attention of many other mathematicians around the world but it also caught the eyes of teachers, researchers and even neuropsychologists who uses the game for diagnosis and treatment purposes. The Hanoi puzzle is a famous one-player problem in recreational mathematics and it is widely used as a teaching example for recursion solution methods in computer science (Houston and Masum, 2004). It is these various applications of the game that inspires me to write this dissertation on the Horizon Tower which is another modification of this well-known mathematical game.

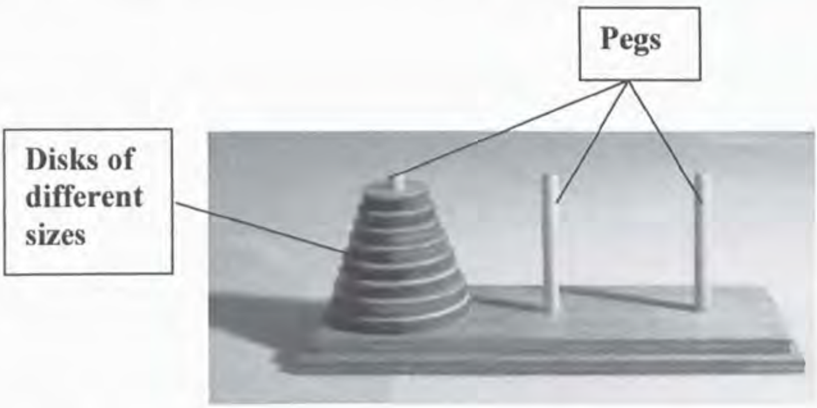

Figure 1.1 A model of the Tower of Hanoi game (Wikipedia(b), 2006).

1.1.1 Rules of The Tower of Hanoi Game

The objective of playing the Tower of Hanoi game is to move an entire stack of *n* disks from one peg to another peg within the fewest moves possible by complying with certain rules. These unique rules of the Tower of Hanoi game make the game very interesting and mind-challenging. It is with these same rules that the investigation of the modified Tower of Hanoi is conducted. The following are the conditions and rules that needs to be complied with when playing the game (Wikipedia(a), 2006):

i. Only one disk can be moved at one time.
ii. No larger disk can be placed on a smaller disk.
iii. The moved disk must the topmost disk.

1.2 Preface to the Horizon Tower

Named as "The Horizon Tower", this variant of the Hanoi game has two towers on one pole or peg. The first tower, also known as the lower or inverse tower, has its disks arranged facing downwards (inverse of a tower) while the second tower or also known as the upper tower, has its disks arranged facing upwards like the Tower of Hanoi and is placed on the first tower. The Horizon Tower, like the Tower of Hanoi, consists of three pegs with disks of different sizes and it differs from the Tower of Hanoi by its arrangement (Figure 1.3).

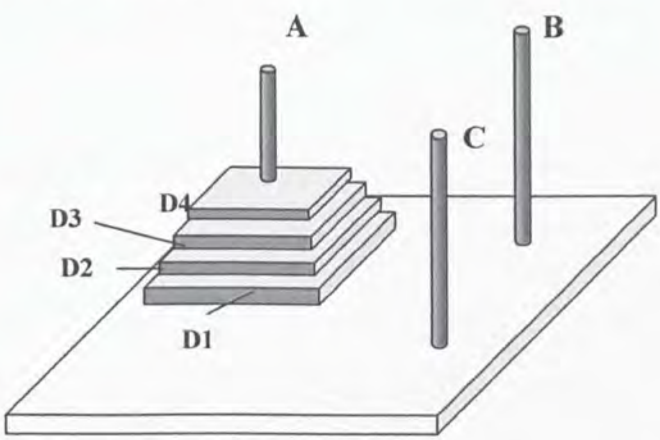

Figure 1.2 The Tower of Hanoi with four disks.

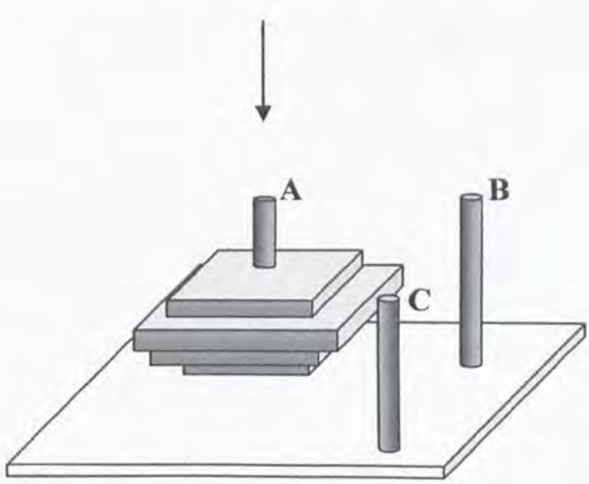

Figure 1.3 The Horizon Tower with four disks.

However, the arrangement above is only an example and it is not the only way of arranging the Horizon Tower. There is more than one way to arrange the tower which is discussed in this dissertation. The divine rule – no larger disk can be placed on a smaller disk – is not obeyed when the tower is rearranged but the rule has to be obeyed when the game is played. The Horizon tower also differs from the original game by colours. The Tower of Hanoi has only one colour but the Horizon Tower has two colours (blue and red). The rules of this modified game is the same as the original, the objective however, is to move the disks into another peg to form either one tower or to move the disks into the other two pegs to form two towers that is of the same colour.

Calling the game "The Horizon Tower" is actually an inspiration from my name – Navaneesha. Navaneesha is a Sanskrit name where Neesha means horizon, furthermore, from the 4-disk Horizon Tower in Figure 1.4, it is evident that the largest disk (D1) acts as the separator of the upper and the lower part of the tower. D1 is just like the horizon that separates the sky and the sea, hence, the name "The Horizon Tower". Also, due to the special characteristics of the largest disk (D1), D1 is given the title – Horizon Disk.

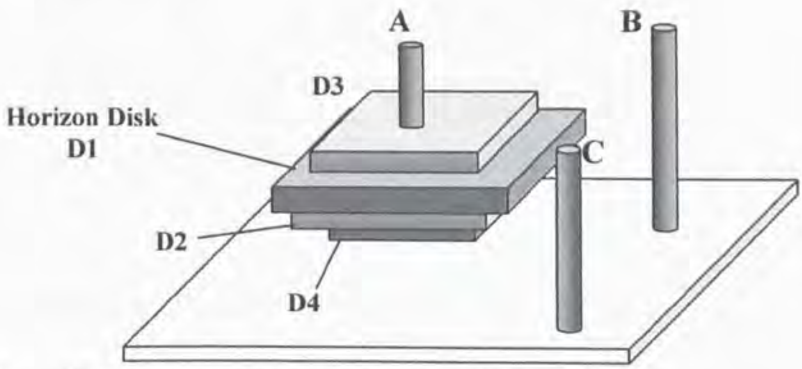

Figure 1.4 The coloured Horizon Tower with four disks.

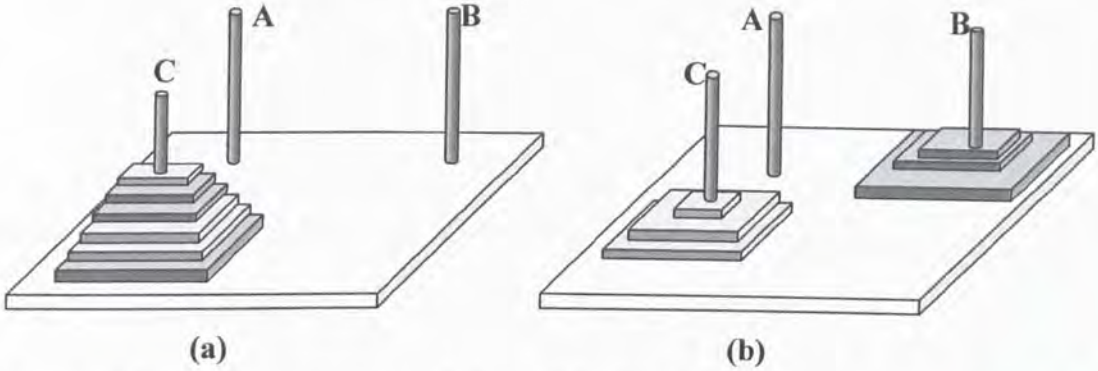

Figure 1.5 (a) The one tower solution while **(b)** The answer to solve the puzzle to form two towers according to their colours.

1.3 Objectives of Research

i. To find a general mathematical formula for the number of different methods of arranging the Horizon tower.

There are many ways of arranging disks to form towers but when given *n* number of disks, knowing how many arrangements are there to form the Horizon Tower is important to make the game interesting. Therefore, if a general mathematical formula is obtained, it will help reduce the burden of manually calculating the number possible arrangements available.

ii. To compare the original Hanoi graphs for the Tower of Hanoi problem with the Hanoi graph for the Horizon Tower.

The Hanoi graphs are very important in evaluating the Tower of Hanoi, therefore how would the graphs changed when the Horizon Tower is played? Since there is a difference of arrangement in the disk, it is interesting to know how these changes will affect the graphs. Hence, comparing the graphs is one of the objectives of this dissertation.

iii. To find the minimum number of moves (optimal solution) to solve this modified game.

If these rearranged towers can be solved, the next question would be if the solution found is optimal or is the solution the minimum number of moves required to solve the game? Thus another objective of this research is to find the minimum number moves. Here the method of the Hanoi graph will be used to evaluate the solutions obtained.

iv. To find out how the disks should be moved to obtain the optimal solution.

How the disks should be transferred from one peg to another is very important. One wrong move will lead to the number of moves not being optimal. Hence it is essential to know how the disks should be transferred. Again the Hanoi graph will be used to determine the best movements of the disks.

1.4 Scope of Research

To ensure that the span of this research is not too wide, there are a few constraints involved. The following is the constraint or scope of research:

i. This game contains only two colours and only three pegs are used.

ii. The disks are rearranged in such a way the number of disks shaped like the Tower of Hanoi (including the Horizon Disk) and the number of disks shaped like the inverse (D2 in Figure 1.6 and in Figure 1.7, D2 and D4) of a tower cannot differ by more than one disk which means the difference can be either one or zero (Figure 1.6 and Figure 1.7).

iii. The position of the solved tower or towers should not be on the original peg, it has to be on peg B or peg C and not on peg A.

iv. The maximum number of disks being discussed is four where the Horizon Disk is given the colour red.

v. The Hanoi Graphs for the Horizon Tower must be shaped like a triangle where each corner represents all the disks being on one peg.

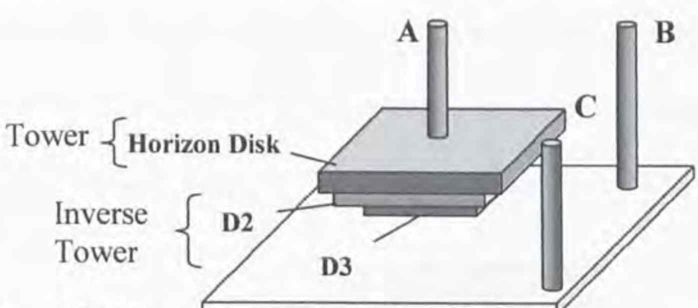

Figure 1.6 The Horizon tower with three disks.

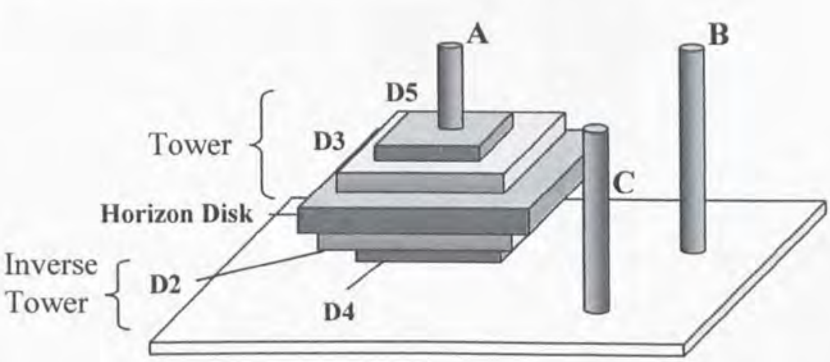

Figure 1.7 An example of the Horizon tower with more than three disks.

CHAPTER 2

LITERATURE REVIEW

2.1 History

The Tower of Hanoi is sometimes referred to as the Towers of Hanoi, Tower of Brahma or the End of the World Puzzle (Lawrence Hall of Science, 2004). It is a famous mathematical puzzle or game invented by a French mathematician, François Edouard Anatole Lucas. Early versions of it carried the name "Prof. Claus" of the College of "Li-Sou-Stain," but these were quickly discovered to be anagrams for "Prof. Lucas" of the College of "Saint Louis" (Darling, 2006).

Figure 2.1 Professor Eduoard Lucas (University of St Andrews, 1996).

Invented in the year 1883, this game consist of three pegs, and on one peg, eight disks of different sizes are arranged. The disks are arranged in an increasing size, from the smallest at the top to the largest at the bottom. The ultimate goal of the game is to move all the disks to another peg, placed in the same order. As mentioned in Chapter 1, in this game, the player could only move one disk at a time but the moves are subjected to the divine rule: to never have a larger disk above a smaller disk. It was introduced as a variant of the mythical "Tower of Brahma", which had real towers and 64 golden rings (Dinitz and Solomon, 2006).

Using recursive methods, the minimum moves required to solve the puzzle are calculated using the formula 2^n-1, where this is an optimal solution to the puzzle. The first solution to the Tower of Hanoi published in the mathematical literature appeared in 1884 in an article by Allardice and Fraser (cited in Lee, 2006).

Figure 2.2 The Tower of Hanoi game being played.

To prove the formula, let there be k+1 disk on one peg. Leave the largest alone and shift the k other disks to another peg. Since any disk can be placed on the largest disk, the transfer of the k smaller disks can be achieved in 2^k-1 moves. Now move the largest disk to the vacant pole onto the largest disk and finally move the k smaller disks back onto the largest disc in 2^k-1 more moves. The total number of moves required is $(2^k - 1) + 1 + (2^k - 1) = 2^{k+1} - 1$. It therefore follows from mathematical induction that the problem can always be resolved in 2^n-1 moves (Gilbert and Vanston, 2005). Another way to proof that the solution is minimal is by using the Hanoi graph (Weisstein, 1996).

The first move required to solve the game can be easily known just by knowing the number of disks involved. If the number of disks is even, the first disk is moved to peg B and if it is odd, the first disk is moved to peg C (Northrop, 1944). In order to play *n*-disk game, the (*n*-1)-disk game should infect be played first. For an example, to play the 4-disk game, the bottom disk cannot be moved until the three disks above it are moved to another peg. It is only then that the bottom disk can be moved to the required peg where it does not need to be moved anymore, then what is left is to play the three disks game (Higgins, 1998).

The following is the solution or the exact method to the solution of the Tower of Hanoi along with the minimum number of moves (optimal solution) needed to solve the Tower of Hanoi problem from one disk to three disks (http://mathforum.org, 2006).

i. 1 disk: 1 move

Move 1: move disk 1 to post C

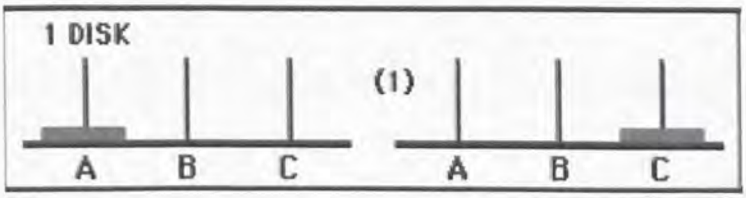

Figure 2.3 The required moves to solve 1-disk puzzle.

REFERENCES

Darling, D., 2006. *Tower of Hanoi*. Retrieved 10 May 2006 from http://daviddarling.info/encyclopedia/T/Tower_of_Hanoi.htm

Dinitz, Y. and Solomon, S., 2006. *Optimality of an Algorithm Solving the Bottleneck Tower of Hanoi problem*. Retrieved 2 Jun 2006, http://www.cs.bgu.ac.il/~dinitz/

Gilbert, W. J. and Vanston, S. A., 2005. *An Introduction to Mathematical Thinking*. Pearson Prentice Hall.

Higgins, P. M., 1998. *Mathematics for the Curious*. Oxford University Press, New York.

Houston, B. and Masum, H., 2004. Exploration in 4-peg Tower of Hanoi. Retrieved 2 Jun 2006 from http://www.scs.carleton.ca/research/tech_reports/2004/TR-04-10.pdf.

Klavzar, S., Milutinovic, U. and Petr, C., 2005. *Hanoi Graphs and some Classical numbers*. Retrieved 9 May 2006 from http://www.computerscienceweb.com

Knoblock, C. A., 1990. *Abstracting the Tower of Hanoi*. Retrieved 19 May 2006 from http:/ /www.cs.cmu.edu/knoblock90-hanoi.pdf

Lawrence Hall of Science, University of California, 2004. *Tower of Hanoi*. Retrieved 9 May 2006 from http://www.lawrencehallofscience.org

Lee, M. K., 2006. *The Graph for the Tower of Hanoi with Four Pegs*. Retrieved 19 May 2006 from http://academic.sun.ac.za/mathed/AMESA/Lee.pdf

Majumdar, A. A. K., 1994. Generalized Multi–Peg Tower of Hanoi Problem. *Journal of the Australian Mathematical Society* **38** (series B), 201-208.

Minsker, S., 2005. *The Little Towers of Antwerpen problem*. Retrieved 19 May 2006 from http://www.sciencedirect.com/sdarticle

Northrop, E. P., 1944. *Riddles in Mathematics, a book of Paradoxes*. English Universities Press.

Odifreddi, P., 2005. *Recursive Functions.* Retrieved 2 Jun 2006 from http://plato.stanf ord.edu/entries/recursive-functions

Snapp, R. R., 2005. *Tower of Hanoi*. Retrieved 9 May 2006 from http://www.cs.edu/~ snapp/teaching/CS5/lectures/hanoi.pdf

Tower of Hanoi (2006). Retrieved 10 Jun 2006 from http://mathforum.org.

University of St Andrews, Scotland, 1996. *Francois Edouard Anatole Lucas*. Retrieved 2 Jun 2006 from http://www-history.mcs.st-andrews.ac.uk/Biograph ies/Lucas.html

Wein, R., 2006. *Hanoi.C*. Retrieved 9 May 2006 from http://www.cs.tau.ac.il/~wein/ courses/C/hanoi.html

Weisstein, E. W., 1996. *Tower of Hanoi*. Retrieved 10 May 2006, http://mathworld.w olfram.com/TowerofHanoi.html

Wikipedia(a), 2006. *Tower of Hanoi*. Retrieved 10 May 2006 from http://www.referen ce.com/browse/wiki/Tower_of_Hanoi

Wikipedia(b), 2006. *Tower of Hanoi*. Retrieved 19 May 2006 from http://experts.about.c om/e/t/to/tower_of_hanoi.htm

Wu, J. S. and Wang, Y. K., 2003. *An Optimal Algorithm to Implement the Hanoi Towers with parallel moves*. Retrieved 10 May 2006 from http://www.science direct.com