COMPARISON BETWEEN GRAPH METHOD AND MATRIX METHOD PLAYED ON NIM GAME

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ABSTRACT

A research was conducted to study the difference between two methods used to play Nim game. The Nim game is a famous take away game, also commonly known as two-player, zero-sum finite game. There are two methods used in this study namely graph method and matrix method. The concentration of normal play Nim was highlighted. There are two primary problems created to determine the outcomes of the game, firstly Nim with pile sizes in tens and secondly Nim with pile sizes in hundreds. Both methods can be used to represent and solve the Nim problem; however the matrix method was found to be more effective as it can produce the outcomes in a shorter time. The outcomes also showed that there are differences between the two methods. In term of time evaluation, this study showed that the matrix method was more ideal if compared with graph method for solving the Nim problem. Hence, the matrix method is suggested in solving the Nim problem.

ABSTRAK

Satu kajian telah dijalankan bagi mengkaji perbezaan di antara dua kaedah yang berlainan untuk bermain permainan Nim. Nim merupakan sejenis permainan dengan informasi lengkap, permainan berhingga tanpa kesempatan pengembalian posisi di antara dua orang pemain. Dalam hal ini, permainan Nim diaplikasikan dengan menggunakan dua jenis kaedah, iaitu kaedah graf dan kaedah matriks. Penumpuan dalam kes permainan Nim secara normal diutamakan supaya pemain pertama memiliki strategi untuk menang. Dengan itu, dua masalah telah ditimbulkan bagi menyelesaikan masalah Nim, iaitu kes pertama mewakili saiz longgokan objek dalam puluhan, manakala kes kedua pula memerihalkan saiz longgokan objek dalam rantusan. Kedua-dua masalah tersebut dapat diwakili dalam bentuk graf dan matriks, dan penyelesaian yang lengkap dapat diperolehi. Bagaimanapun, kaedah graf menggunakan kaedah jalan kerja yang berlainan dengan kaedah matriks. Namun begitu, kaedah matriks dianggap sebagai cara yang paling berkesan untuk menyelesaikan masalah Nim dalam jangka masa yang pendek. Oleh itu, kaedah matriks dicadangkan sebagai kaedah yang efektif untuk untuk menyelesaikan masalah Nim.

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LIST OF SYMBOLS

XOR	Exclusive OR
$\oplus$	Nim-sum
V	Vertices
E	Edges
$V(G)$	The set of vertices of graph G
$E(G)$	The set of edges of graph G
w	Weight of a graph
$w(e)$	The weight of e
N	Natural numbers
$Mex(S)$	Minimum excluded value of S
a_{ij}	The element of a matrix
α	The golden ratio of Wythoff's Game, $(1+\sqrt{5})/2$
Δ	Piece (starting position for the player) in graph

CHAPTER 1

INTRODUCTION

1.1 Introduction

Games between two players, are the kind of games where one player wins and one loses, it has become a familiar tool in many branches of logic during the second half of the twentieth century (Blass, 1992).

There are close links between games and logics. A game theory is a hybrid branch of applied mathematics and economics that studies strategic situations where players choose different actions in an attempt to maximize their returns (Fudenberg *et al.*, 1991).

The games studied by game theory are well-defined mathematical objects. A game consists of a set of players, a set of strategies available to those players, and a specification of payoffs for each combination of strategies (Fraenkel, 1978).

One of the classical and simple games that applied using mathematics and logic is the game of Nim. A variety of Nim type games can be played by several methods, such as by using logic gate method, graph method, programming method, matrix method and etc (Xinyu, 2005).

For this research, two methods are employed to solve the games; there are graph and matrix method. Then the differences between the both methods will be determined.

1.2 The History of Nim Game

The games of Nim have been played since ancient time. This game is originated from China (similar to the Chinese game of *Tsyanshidzi*, means "picking stones game"), (Domoryad, 1964). In China, the children play it with bits of paper, while adults can be found playing it with coins on the counter of a bar. Nowadays the game of Nim is played by using the different types of objects (counters, coins, tokens, sticks, matches).

Nim was given its name by Charles Leonard Bouton who named it after an archaic English word meaning to steal or to take away or from the German verb *nimm* (meaning "take!"), (Gardner, 1983). The theory of the Nim game was discovered by the mathematics professor Charles Leonard Bouton at Harvard University in 1901. The full analysis and proof containing a winning strategy for Nim is published. The strategy was based on the nim-sum, or bitwise XOR (exclusive-OR), of the pile sizes.

Wythoff's introduced the game of Nim in 1907 and gave the formula for its P-positions (Wythoff, 1907). It was reinvented by the Dutch mathematician Willem Abraham Wythoff (1865-1939), (the Dutch spelling is *Wijthoff*), who published its full analysis in 1907. Half a century later (around 1960) and unaware of this, the mathematician Rufus P. Isaacs of Johns Hopkins University gave another description of the game (Gardner, 1983).

The theory of impartial games was presented by Grundy P. M. in 1939. Later, it was realized that Sprague Roland P. had published the theory independently in 1935. Therefore, now it is known as the Sprague-Grundy Theory. This theory is related to the impartial game such as Nim.

Curiously, Alain Resnais featured the Nim game in the movie *"L'année dernière à Marienbad"* (Last Year in Marienbad, 1962) starring by Delphine Seyrig, Giorgio Albertazzi, and Sacha Pitoëff.

Figure 1.1 The Movie of Last Year At Marienbad (Bouton, 1902)

The "classic" Nim Game is a game by two players. It consists of 16 matches in four rows as shown below. Two players alternately pick a certain number of matches and the one, who takes the last match, wins the game (Moore, 1910).

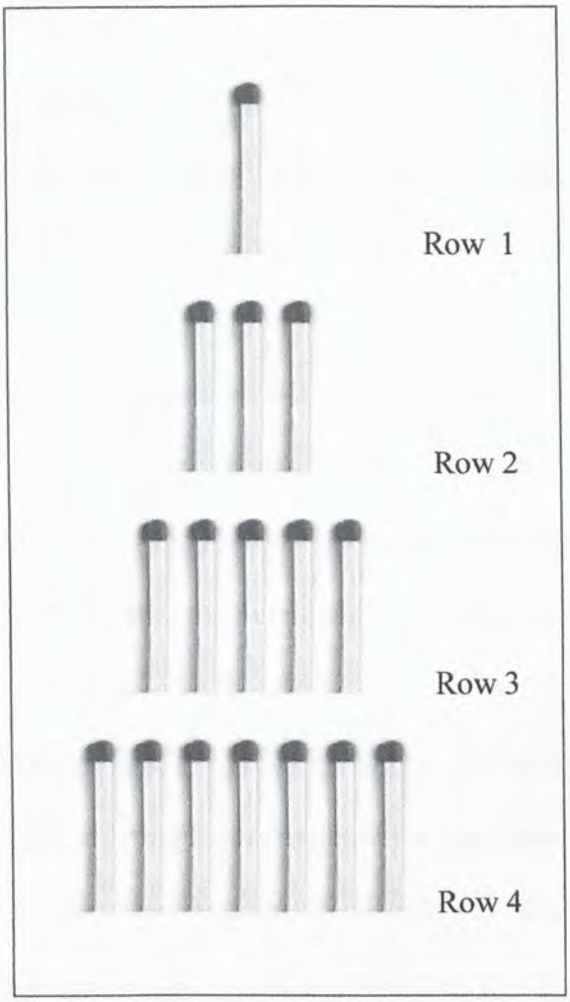

Figure 1.2 The classical of Nim game consists of 16 matches in four rows

There are many games derived from, or closely related to Nim. Two of the most famous ones are Tic-Tac-Toe and match 23. The Nim game is a great game to play, it encourages logical thinking and a fun way to think mathematically. So, the variations of Nim make it so challenging for all ages (Weston, 2005).

1.3 What is Nim Game?

The well-known game of Nim is a two player, zero-sum finite game. According to Bouton, (1901), the initial setup for a play of Nim consists of three separate piles of sticks; each of pile has a finite size. A turn consists of selecting one pile and then removing any nonzero number of sticks from that pile, leaving the other two piles alone. Two players take alternate turns until all sticks are removed. The player that picks up the last stick is the winner (Oltean, 2001).

Nim also a two-player mathematical game of strategy in which players take turns removing objects from distinct heaps. On each turn, a player must remove at least one object, and may remove any number of objects provided from the same heap. The winner is the player who takes the last stick (Yaglom *et al*., 1967).

Nim is a two-player game, which one of the players always has a winning strategy means a way of choosing the moves so as to guarantee a win no matter the moves of the other player. The standard approaches for determining the winning strategies for Nim are based on the Sprague-Grundy theory (Berlekamp *et al*., 1982). The first winning strategy for the Nim game was proposed by Charles Leonard Bouton from the Harvard University. The Bouton's solution is based on computing the Nim-sum by using XOR (exclusive-OR) operation of the numbers of objects in each heap. The strategy of playing Nim will be discussed later.

1.4 Types of Nim Game

Nim can be played as misère game and normal game. Misère games are played by the same rules as the normal ones with one distinguished exception: while in the normal game the player unable to move loses the game, in the misère games, the player unable to move wins the game (McCain, 2004).

In the misère game, the players are presented with one or more piles (or heaps) of objects (coins, counters, stones). A move consists in removing a number of objects from a single pile. The winner is the last person who removes the last object in the single heap. Normally, the misère games are far more difficult than the normal play Nim (Ferguson, 1984).

Normal play Nim (or more precisely the system of nimbers) is fundamental to the Sprague-Grundy theorem, which can say that in normal play every impartial game is equivalent to a Nim heap that yields the same outcome when played in parallel with other normal play impartial games (Fraenkel *et al.*, 2001).

The Sprague-Grundy theorem states that every impartial game is equivalent to a nimber. The nim-value of an impartial game is then defined as the unique nimber that the game is equivalent to (Dress *et al.*, 1999).

In the present study, the normal play convention of Nim is considered, which means that the player who makes the last move wins the game. The purpose of normal play Nim is chosen because most of games follow this kind of convention, that is who makes the last move wins the game.

The other reason normal play Nim is chosen because of the purposes of the Sprague-Grundy theorem. According to this theorem, an impartial such as Nim, is a two-player game in which both players have complete information, no chance is involved and the legal moves from each position are the same for both players. One of the both players always has a winning strategy (Fraenkel *et al.*, 1986).

1.5 How to Play Nim?

Nim is a game that plays with logics and mathematical theory. Nim has been mathematically solved for any number of initial heaps and objects. Nim starts with a group of heaps, each of which contains one or more sticks from a single heap until all sticks are removed from the heaps. The player that picks up the last stick is the winner.

Here are the general ways how to play the Nim game. At first, draw the following sticks with a pencil on a paper. The figure is as below:

1 1 1 1 1

1 1 1 1

1 1 1

1 1

1

Figure 1.3 Nim game with several rows

Then, two players take alternate turns to remove the stick. On each turn a player crosses out marks on one row of stick only. The player may cross out as few or many as wishes, but must cross out at least one. The player who crosses out the last stick considered as a winner (Schaefer, 1978).

Notice that position on a line does not matter. For example, if the player going first wishes to take away two sticks from line three, it is no different whether the player takes away the two on the right of row three or the two on the left of row three. Thus, for ease of play it is traditional to take away sticks from the right. The player would cross out marks indicated by x's.

```
1   1   1   1   1
  1   1   1   1
    1   x   x
      1   1
        1
```

Figure 1.4 The cross out marks of the sticks indicated by x's

In the present study, graph and matrix method will be used to play Nim. Now, briefly review how to play the Nim on graph and matrix. For the matrix method, the position is the rectangular array, say *m* by *n* of piles of sticks. While for the graph method, it denoted by undirected and finite graph with a set of vertices and a set if edges.

1.6 The Strategy of Playing Nim

Nim is a two-player game played with sticks. The sticks are divided into piles. The players take alternate turns to remove the sticks. On each player's turn, the player may remove any number of sticks from one of the pile, but can only take from a single pile on a given turn.

The goal is to take the last stick. Whoever takes the last stick wins. A winning strategy is needed to win the game. So, with the multiple heaps and the possibility to remove as many sticks that is wanted from either one of the heaps, then the Nim-sum is compute, which characterizes the configuration of the game. The nim-sum of x and y is written as $x \oplus y$ to distinguish it from the ordinary sum, $x + y$.

Nim-sum is based on Boolean arithmetic and, in addition, a binary representation of integers. The nim-sum is compute by using exclusive OR (XOR) operation. The logic table for XOR is as bellows:

Table 1.1 The logic gate for XOR

Input 1	Input 2	Output
1	1	0
1	0	1
0	1	1
0	0	0

Here is an example of the calculation with heaps of size 3, 4 and 5 is as follows:

Table 1.2 Compute the nim-sum in binary and decimal form

Binary	Decimal	
011_2	3_{10}	Heap A
100_2	4_{10}	Heap B
101_2	5_{10}	Heap C

The nim-sum of heaps A, B and C, $011_2 \oplus 100_2 \oplus 101_2 = 010_2$ in binary systems, while in the decimal forms, $3 \oplus 4 \oplus 5 = 2$.

In normal play Nim, the winning strategy is to remove all the sticks with a nim-sum of zero, which is always possible if the nim-sum is not zero before the move. If the nim-sum is zero, then the next player will lose if the other player does not make a mistake. To find out how many move to make, let X be the nim-sum of all the heap size. Take the nim-sum of each of the heap size wit X, and find a heap whose size decreases. The winning strategy is to play in such a heap, reducing that heap to the nim-sum of its original size with X.

From the example above, the nim-sum of the size is $X = 3 \oplus 4 \oplus 5 = 2$. The nim-sums of the heap sizes $A = 3$, $B = 4$ and $C = 5$ with $X = 2$ are:

$$A \oplus X = 3 \oplus 2 = 1$$

$$B \oplus X = 4 \oplus 2 = 6$$

$$C \oplus X = 5 \oplus 2 = 7$$

The only heap that is reduced is heap A, so the winning move is to reduce the size of heap A to 1 by removing two objects. If there are only two heaps left, the strategy is to reduce the number of objects in the bigger heap to make the heaps equal. After that, no matter what move the opponent makes, the same move still can be made on the other heap, guaranteeing that the last stick is taken.

As a conclusion, denote the game by (*A, B, ..., K*) where there are heaps of objects of size *A, B, ..., K*. The nim-sum of numbers *A, B, ..., K*, is written as $A +_2 B +_2 ... +_2 K$ is add the numbers in binary without carrying. For example, Nim with heaps of size 3, 5, and 7 is equivalent to a heap of size $3 +_2 5 +_2 7 = 1$. It means that each of the heaps of size is represented in binary representation of integers. So, the winning strategy for normal play Nim is $A +_2 B +_2 ... +_2 K = 0$, where the nim-sum is zero and guarantee the first player whose start the game will win the game by making the last moves.

1.7 Objectives of Research

- To represent and solve the Nim game by graph and matrix method
- To compare the differences between graph and matrix method
- To determine which method is the effective way to solve the game

1.8 Research Scope

In this research, the normal play Nim is considered only, which means that the player who makes the last move wins the game. It is because most games follow this kind of convention. The normal play rule is fundamental to the Sprague-Grundy theorem that is easy to compute the normal Nim. The Nim game is represented in graph and matrix method in addition to compute and solve the Nim by using two main problems. In graph method, some finite undirected graph will be used to solve the Nim and assigned a non-negative integer to each edge that represent a piles of sticks, each of which contains one or more sticks. While in the matrix method, the pile of sticks will

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