SOLVING FUZZY SYSTEM OF LINEAR EQUATION USING CONJUGATE GRADIENT METHOD

HAW HUI HOE

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DECLARATION

I declare that this writing is my own work except for citations and summaries where the sources of every each have been duly acknowledged.

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HAW HUI HOE
HS 2004-1032

AUTHENTICATION

Signature

1. **SUPERVISOR**

   (Assoc. Prof. Dr. Jumat Sulaiman)

2. **EXAMINER**

   (Miss Suzelawati Zenian)

3. **DEAN**

   (SUPT/KS Assoc. Prof. Dr. Shariff A. K. Omang)

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ABSTRAK

Menyelesaikan Sistem Persamaan Linear Kabur
Menggunakan Kaedah Kecerunan
Konjugat

Terdapat pelbagai ketidakpastian dalam menukarkan ukuran dalam masalah harian kepada bentuk ukuran berunsurkan nombor. Jadi, sistem persamaan linear kabur diperkenalkan untuk mengatasi masalah ini. Demi menyelesaikan sistem persamaan kabur dengan lebih mudah dan cepat, kaedah berangka digunakan. Tajuk disertasi ini adalah menyelesaikan masalah sistem persamaan linear kabur menggunakan teorem Kecerunan Konjugat. Dalam disertasi ini, kaedah Kecerunan Konjugat digunakan untuk menyelesaikan sistem persamaan linear kabur yang berbentuk $m \times n$ dan dibandingkan dengan kaedah Gauss-Seidel serta kaedah SOR. Aspek yang akan dikaji dalam perbandingan antara ketiga-tiga kaedah ini ialah jumlah lelaran, masa yang diambil untuk mendapatkan penyelesaian dan ralat maksimum.

ABSTRACT

There are uncertainties in transforming real life measurements into numbers. Therefore, fuzzy system of linear equation (FSLE) is getting more and more common to tackle these uncertainties. These linear systems can be solved easily using numerical methods. In this dissertation, the Conjugate Gradient (CG) method is being used to solve rectangular FSLE and to be compared to the classical Gauss–Seidel (GS) and SOR methods. The aspects to be compared are number of iterations, computational time and maximum error between these two methods.

CONTENT

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

$=$	equal to
$\neq$	not equal to
Min	minimum
Max	maximum
$<$	less than
$>$	more than
$\leq$	less or equal to
$\geq$	more or equal to
ε	error tolerance
∇	gradient
∂	partial differentiation
T	transpose matrix
L	lower matrix
U	upper matrix
I	identity matrix
D	diagonal matrix
Det	determinant
$\sum$	summation
$\in$	element of
$\rightarrow$	implies that

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION TO ARTIFICIAL INTELLIGENCE

The origin of Artificial Intelligence (AI) and the concept of intelligent machines may be found in Greek mythology where the myths antiquity involves human-like artifacts. In the 5^{th} century B. C., Aristotle invented syllogistic logic, the first formal reasoning system. Many other basic machines have been invented from the 13^{th} to 19^{th} century which includes cloaks. At the same time, George Boole developed a binary algebra representing "laws of thought" (Buchanan, 2002).

The term 'Artificial intelligence' (AI) was first introduced in 1956 by John McCarthy in Dartmouth conference. Instead of concentrating on the hardware to imitate intelligence, the meeting set the course for examining the structure of the data being processed by computers, the use of computers to process symbols, the need for new languages and the role of computers for testing theories. AI may be defined as the branch of computer science that is concerned with the automation of intelligent behaviors. AI is a field of study that encompasses computational techniques for performing tasks that require human intelligence. It is a technology of information processing concerning with process of reasoning, learning and perception. The branches of AI are (Badiru & Cheung, 2002):

i) Natural language processing deals with various areas of research such as database inquiry systems, story understanders, automatic text indexing, grammar and style analysis of text, automatic text generation, machine translation, speech analysis and speech synthesis.

ii) Computer vision deals with research efforts which involved scene analysis, image understanding and motion derivation.

iii) Robotics involves the control of effectors on robots to manipulate or grasp objects, locomotion of independent machines and use of sensory input to guide actions.

iv) Problem-solving and planning involves applications like refinement of high-level goals into lower-level ones, determination of actions needed to achieve goals, revision of plans based on intermediate results and focused search of important goals.

v) Learning deals with research into various forms of learning including rote learning, learning through advice, learning by example, learning by task performance and learning by following concepts.

vi) Expert systems deal with the processing of knowledge as opposed to the processing of data. It involves the development of computer software to solve complex decision problems.

For many practical systems, important information comes from two sources:

i) Human experts who describe their knowledge about the system in natural languages.

ii) Sensory measurements and mathematical models that are derived according to physical laws.

An important task is to combine these two types of information into system design and fuzzy system will be used to perform the transformation from a human knowledge base into a mathematical formula to develop expert system.

1.2 FUZZY MATHEMATICS

There are many common techniques available for handling uncertainty in expert systems in AI. Expert systems consultations for practical problems often require some simplifying assumptions to be made. An approach to manage uncertainty is the concept of fuzzy sets (Tomsovic, 2006).

Fuzzy sets were first introduced by Lotfi A. Zadeh in 1965 and the objective was to generalize the notion of a set and propositions to accommodate the type of fuzziness or vagueness in many decision problems. Uncertainty in fuzzy logic typically arises in the form of vagueness and/or conflicts, which are not represented naturally within the probabilistic framework. In formal truth logic, it is required that every proposition be either true (1) or false (0). "0" and "1" fits conventional computers processes perfectly but it can impose serious restriction on machine reasoning intended to duplicate the imprecise aspects of human reasoning. Statements that contain uncertainty are all imprecise and may require clarification. Fuzzy sets emphasize the modeling of such uncertainties (Tomsovic, 2002).

While there are many computational algorithms developed to process numerical data, fuzzy system provide an alternative way to manipulate information, not just data in AI. Fuzzy systems are one of the applications at the leading edge of AI. Fuzzy systems can be found in expert system and are knowledge-based or rule-based systems constructed from a collection of fuzzy IF-THEN rules provide an alternative approach to represent problems and process information (Badiru & Cheung, 2002).

Other than in AI, fuzzy systems play a major role in various areas such as economics and finance, physics, statistics and engineering. In many applications, at least some of the system's parameters are represented by fuzzy quantities rather than crisp numbers. Some of the most useful capabilities and features provided by modeling in fuzzy set approaches are: (Tomsovic, 2006)

i) Representation methods for natural language statements,
ii) Models of uncertainty where statistic are unavailable or imprecise,
iii) Information models of subjective statements,
iv) Measures of the quality of subjective statements,
v) Integration between logical and numerical methods,
vi) Models for soft constraints,
vii) Models for resolving multiple conflicting objectives,
viii) Strong mathematical foundation for manipulation of the above representations.

1.3 NUMERICAL METHOD

Many problems in applied mathematics involve solving systems of linear equations with the linear system occurs naturally in some cases and as a part of the solution cases in other cases. Numerical method is the area of mathematics and computer science that creates, analyzes and implements algorithms for solving numerically the problems of continuous mathematics.

There are two common types of solution methods to solve systems of linear equation

$$Ax = b \tag{1.1}$$

which are direct and iterative methods. Direct methods lead to a theoretically exact solution x in a finite number of steps. Iterative method is approximate method which creates a sequence of approximating solutions in increasing accuracy manner. Specialized methods have been developed for linear systems with special properties such as symmetric and diagonal.

Many advanced techniques are available to increase savings in time and/or space when solving linear algebraic equations as shown in Figure 1.1. Apart from $n \times n$ sets of equations, there are other systems where the number of equations, m and number of unknowns, n are not equal systems where $m < n$ are called underdetermined and $m > n$ are called overdetermined. For underdetermined matrix, there can be either no solution or else more than one. For overdetermined cases, there is in general no exact solution. However it is possible to develop a compromise solution that attempts to determine answers that come "closest" to satisfy all the equations simultaneously (Woo, 2004).

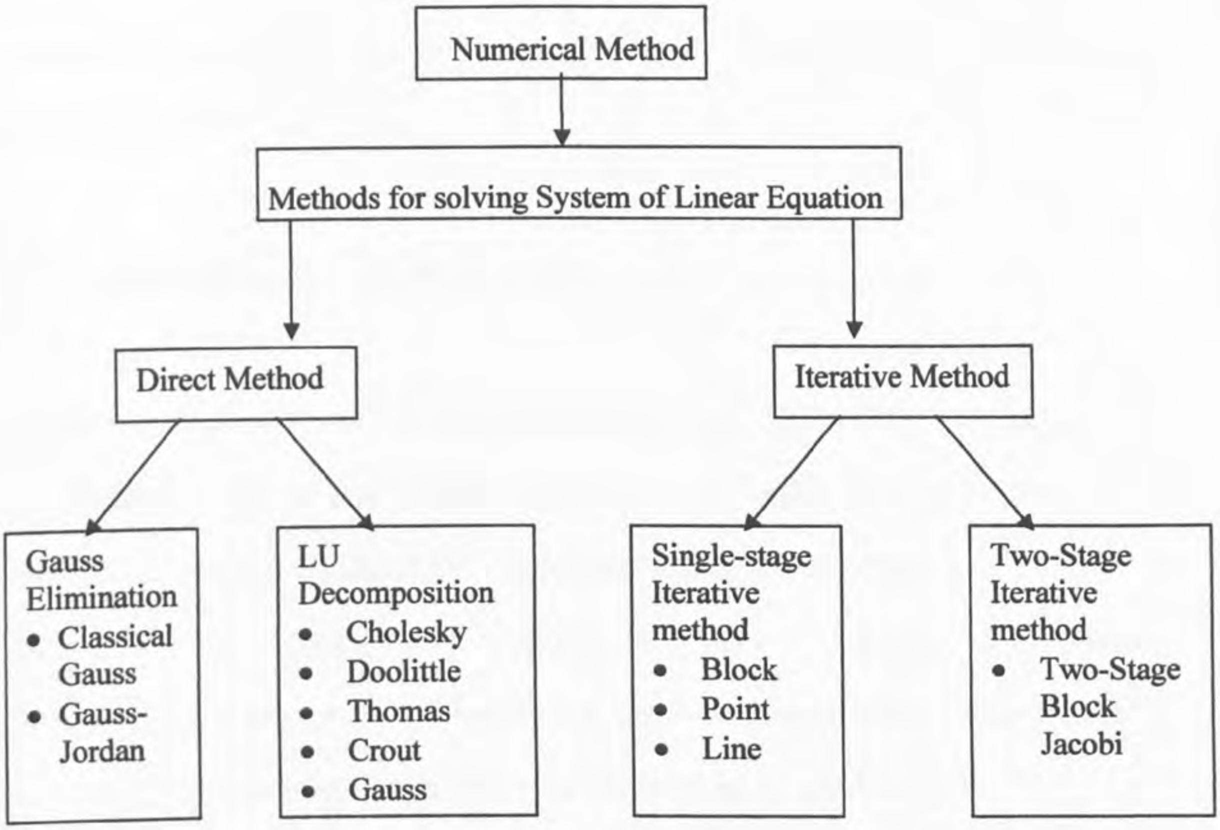

Figure 1.1 Classification of common methods in solving linear systems.

1.3.1 Gauss-Seidel Method

The basic idea of Gauss-Seidel (GS) method is that a more rapid convergent scheme will be obtained if new results is to be used immediately as soon as they have been known. In this scheme the i^{th} equation is given by

$$x_j^{(m+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(m+1)} - \sum_{j=i+1}^{N} a_{ij}x_j^{(m)}\right) \qquad i = 1,2,3,\ldots N \tag{1.3}$$

By comparing the GS with Jacobi method, it can be shown theoretically that when the two methods converge, Gauss-Seidel method does twice as fast as Jacobi method for typical problems with matrix structure such as sparse, band matrix (McDonough, 2001).

1.3.2 Conjugate Gradient Method

The Conjugate Gradient (CG) method of Hestenes and Stiefel [HS] was originally developed as a direct method designed to solve a $n \times n$ positive definite linear system. As a direct method, it is inferior to Gaussian elimination with pivoting because the CG method is more computationally expensive than those in Gaussian elimination. However, the CG method is very useful when employed as an iterative method (Faires & Burden, 2003). CG is effective for systems in the form of equation (1.1)

$$Ax = b \tag{1.2}$$

where x is an unknown vector, b is a known vector and A is a known, square, symmetric and positive definite (or positive-indefinite) matrix (Shewchuk, 1994). Quasi-minimal residual method, Preconditioned Conjugate Gradient (PCG) method (to improve the condition number of an ill-conditioned matrix), generalized minimum residual method, Biconjugate Gradient method (a generalization of the Conjugate Gradient method for use in solving n-by-n linear system that is not necessarily either symmetric or positive definite) and CG Squared method can be derived from Conjugate Gradient method (Fausett, 2003).

1.4 OBJECTIVE OF RESEARCH

This research is aimed to solve fuzzy system of linear equation (FSLE). This research is focused mainly in solving $m \times n$ rectangular fuzzy system, where $m \neq n$. The method used in this research will be CG method along with classical GS method which acts as control method. The objectives in this research are:

i) Derive and find the approximate solution for $m \times n$ rectangular FSLE ($m \neq n$).

ii) To develop the CG method for solving FSLE.

iii) To verify that numerical solutions of CG method is faster than GS method in terms of number of iterations, execution time and maximum error.

1.5 SCOPE OF RESEARCH

This research will solve $m \times n$ rectangular fuzzy matrix in FSLE where $m \neq n$. However, overdetermined FSLE which cannot be solved is not discussed in this research. In this research, fuzzy system of linear equation will not be derived from real world problem. Only iterative methods which are the CG method and GS methods will be considered.

CHAPTER 2

LITERATURE REVIEW

2.1 FUZZY SYSTEM OF LINEAR EQUATION

According to Tomsovic (2006), fuzzy logic implements experiences and preferences through membership functions. Fuzzy rules may be formed describing relationships linguistically as antecedent-consequent pairs of IF-THEN statements. Basically, there are four approaches to the derivation of fuzzy rules:

i) From expert experience and knowledge,

ii) From the behavior of human operators,

iii) From the fuzzy model of a process,

iv) From learning

A fuzzy set is a set of ordered pairs with each containing an element and the degree of membership for that element. A higher membership value indicates that an element more closely matches the characteristic feature of the set. For fuzzy set A:

$$A = \{(x, \mu_a(x)) \mid x \in X\} \tag{2.1}$$

where X is the universe, $\mu_a(x)$ represents the membership function and for normalized sets $\mu : X \rightarrow [0,1]$ (Tomsovic, 2006).

Other authors who have contributed to fuzzy system of linear equation are Buckley and Qu (1991). They have discussed about finding solutions for $\bar{x}$ to the matrix equation

$$Ax = b \tag{2.2}$$

when the matrix equation $\bar{A}\bar{x} = \bar{b}$ where the $\bar{a}_{ij}$ in $\bar{A}$ and the $\bar{b}_i$ in $\bar{b}$ are triangular fuzzy numbers. They also did a nested family of intervals for each a_{ij} and b_i in which researchers consider only one interval for b_i. They also showed how to put the solutions for x together to obtain fuzzy vector $\bar{x}$. From then on, many researchers have used their paper as reference, for example Abbasbandy *et al.* (2005) and Allahviranloo and Kermani (2006).

2.1.1 Square Fuzzy Linear System

Friedman *et al.* (1998) first published a paper on fuzzy linear systems and their solution. They proposed a general model for solving a $n \times n$ fuzzy linear system. A fuzzy $n \times n$ linear system can be written as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= y_n
\end{aligned} \tag{2.3}$$

where the coefficient matrix $A = (a_{ij})$ is a $n \times n$ crisp matrix and $y_i, i = 1,2,\ldots n$ are fuzzy numbers vector and it is called a fuzzy system of linear equations (FSLE).

A fuzzy number vector $(x_1, x_2, ..., x_n)^T$ is given by

$$x_i = (\underline{x_i}(r),\ \overline{x_i}(r)),$$
$$1 \le i \le n,\ 0 \le r \le 1 \tag{2.4}$$

where $\underline{x_i}(r)$ is the lower bound and $\overline{x_i}(r)$ is the upper bound is called a solution of the fuzzy system if

$$\begin{aligned} \underline{\sum_{j=1}^{n} a_{ij} x_j} &= \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{y_i} \\ \overline{\sum_{j=1}^{n} a_{ij} x_j} &= \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{y_i} \end{aligned} \tag{2.5}$$

if for particular $i = a_{ij} > 0,\ 1 \le j \le n$. And eventually

$$\sum_{j=1}^{n} a_{ij} \underline{x_j} = \underline{y_i},\ \sum_{j=1}^{n} a_{ij} \overline{x_j} = \overline{y_i} \tag{2.6}$$

In order to solve fuzzy system of linear equation, one must solve a $(2n) \times (2n)$ crisp linear system where the right-hand side column is the function vector $(\underline{y_1},\ \underline{y_2},\ ...,\ \underline{y_n},\ \overline{y_1},\ \overline{y_2},\ ...,\ \overline{y_n})^T$. The $(2n) \times (2n)$ linear system is obtained where s_{ij} are determined as follows:

$$\begin{aligned} a_{ij} \ge 0 &\Rightarrow s_{ij} = a_{ij},\ s_{1+n,j} = a_{ij} \\ a_{ij} \le 0 &\Rightarrow s_{i,j+n} = -a_{ij}, s_{i+n,j} = -a_{ij} \end{aligned} \tag{2.7}$$

and any s_{ij} which is not determined is zero. From equation (2.3), the matrix notation (Friedman *et al.*, 1998.) is

$$Sx = y \tag{2.8}$$

2.1.2 Rectangular Fuzzy Linear System

Allahviranloo and Kermani (2006) on the other hand have studied the general $m \times n$ fuzzy linear system using the Embedding method. In their paper, they investigated a $m \times n$ rectangular consistent fuzzy linear system whose coefficient matrix is crisp and the right hand side column is a fuzzy number vector, *y*. They first replaced the original $m \times n$ fuzzy linear system by a $(2m) \times (2n)$ crisp linear system and then approximate solutions are obtained by using pseudo inverse and also orthogonal matrix and least-square method.

They represented an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r),\ \overline{u}(r))$, $0 \le r \le 1$ which satisfies the following requirements:

i) $\underline{u}(r)$ is bounded left continuous non-decreasing function over $[0,1]$.

ii) $\overline{u}(r)$ is bounded left continuous non-decreasing function over $[0,1]$.

iii) $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1$.

A general $m \times n$ rectangular fuzzy linear system can be written in

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= y_1 \\
a_{21}x_2 + a_{22}x_2 + \dots + a_{2n}x_n &= y_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= y_m
\end{aligned}
\tag{2.9}
$$

where the coefficient matrix $A = (a_{ij})$, $1 \le i \le m$, $1 \le j \le n$ is a crisp $m \times n$ matrix and y_i with $i = 1, 2, \dots, m$ are fuzzy numbers vector. An assumption was being made is that rank(A) = n for $(n<m)$ or rank(A) = m for $(m \le n)$.

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