

FOURTH-ORDER EXPLICIT RUNGE-KUTTA AND  
SECOND-ORDER EXPLICIT ADAM-BASHFORTH  
METHODS FOR SOLVING  
NONLINEAR DYNAMIC SYSTEM: FROUDE'S PENDULUM

LIM SHIAW HUONG

THIS DISSERTATION IS SUBMITTED IN PARTIAL  
FULLFILLMENT OF THE REQUIREMENT FOR THE  
DEGREE OF BACHELOR OF SCIENCE WITH HONORS

MATHEMATICS WITH COMPUTER GRAPHIC  
PROGRAMME  
SCHOOL OF SCIENCE AND TECHNOLOGY  
UNIVERSITI MALAYSIA SABAH

April 2008

PERPUSTAKAAN  
UNIVERSITI MALAYSIA SABAH



**UMS**  
UNIVERSITI MALAYSIA SABAH

## UNIVERSITI MALAYSIA SABAH

## BORANG PENGESAHAN STATUS TESIS@

JUDUL: FOURTH-ORDER EXPLICIT RUNGE KUTTA AND SECOND-ORDER EXPLICITADAM BASHFORTH METHODS FOR SOLVING NONLINEAR DYNAMIC SYSTEM : FROUDE'S PENDULUMIJAZAH: SARJANA MUDA SAINS DENGAN KEPUJIANSAYA LIM SHIAN HUONGSESI PENGAJIAN: 2005 - 2008

(HURUF BESAR)

mengaku membenarkan tesis (LPSM/Sarjana/Doktor Falsafah) ini disimpan di Perpustakaan Universiti Malaysia Sabah dengan syarat-syarat kegunaan seperti berikut:-

1. Tesis adalah hakmilik Universiti Malaysia Sabah.
2. Perpustakaan Universiti Malaysia Sabah dibenarkan membuat salinan untuk tujuan pengajian sahaja.
3. Perpustakaan dibenarkan membuat salinan tesis ini sebagai bahan pertukaran antara institutsi pengajian tinggi.
4. Sila tandakan (/)

 SULIT

(Mengandungi maklumat yang berdarjah keselamatan atau Kepentingan Malaysia seperti yang termaktub di dalam AKTA RAHSIA RASMI 1972)

 TERHAD

(Mengandungi maklumat TERHAD yang telah ditentukan oleh organisasi/badan di mana penyelidikan dijalankan)

 TIDAK TERHAD

Disahkan Oleh

  
(TANDATANGAN PENULIS)

(TANDATANGAN PUSTAKAWAN)

Alamat Tetap: 2A, SELASIH TIMUR 1,  
JALAN HULU LANANG,  
96000 SIBU SARAWAK

CIK KHADIZAH GHAZALI

Nama Penyelia

Tarikh: 24-04-08Tarikh: 24-04-08

CATATAN:- \*Potong yang tidak berkenaan.

\*\*Jika tesis ini SULIT atau TERHAD, sila lampirkan surat daripada pihak berkuasa /organisasi berkenaan dengan menyatakan sekali sebab dan tempoh tesis ini perlu dikelaskan sebagai SULIT dan TERHAD.

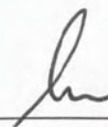
@Tesis dimaksudkan sebagai tesis bagi Ijazah Doktor Falsafah dan Sarjana secara penyelidikan atau disertai bagi pengajian secara kerja kursus dan Laporan Projek Sarjana Muda (LPSM).



## DECLARATION

I declare that this writing is my own work except for citations and summaries where the sources of every each have been duly acknowledged.

30<sup>th</sup> April 2008



---

LIM SHIAW HUONG

HS2005-4483

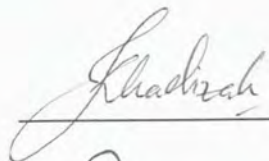


## AUTHENTICATION

Signature

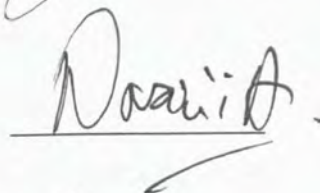
## 1. SUPERVISOR

(Miss Khadizah Ghazali)



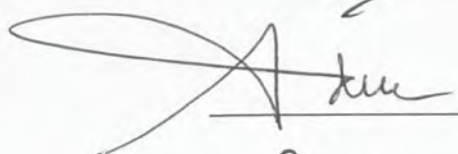
## 2. CO-SUPERVISOR

(Pn. Noraini Abdullah)




## 3. EXAMINER 1

(Prof. Dr. Zainodin Hj. Jubok)



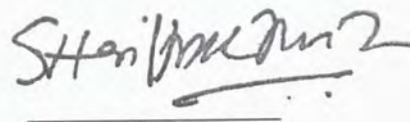
## 4. EXAMINER 2

(Pn. Suzelawati Zenian)



## 5. DEAN

(SUPT/KS Assoc Prof. Dr. Shariff A. K. Omang)



## ACKNOWLEDGEMENT

Out of the entire first, I would like to take this opportunity to thank to my supervisor, Miss Khadizah Ghazali and my co-supervisor, Pn. Noraini Abdullah, for their guidance, advices, support and a lot of help throughout the process of writing this dissertation. All their efforts and sacrifice in time to help me to eliminate all the doubts and problems. I really appreciated it and would not be forgotten. Beside, I also thanked mathematic lecturers that are Assoc. Prof. Dr. Jumat Sulaiman and Dr Aini Janteng who have provided me with some fundamental concepts of mathematic in this dissertation.

I would also like to thank the staff of School of Science and Technology who always arranging the process for final year project and also their assistance in completing this dissertation.

Not forgetting my appreciation towards my fellow friends who have provided me valuable thoughts, ideas throughout the process of writing this dissertation. Besides, they always give me support when I get frustrated in the writing process. My fellow friends also help me to correct the mistakes occurred and offend some usable idea which can help me to discover the imperfectness of this dissertation.

Finally, I would like to thank my lovely family, for their encouragements and support throughout the writing process. Thanks to all.



## ABSTRAK

Dalam kertas kerja ini, kaedah Runge Kutta peringkat ke-4 dan kaedah Adam Bashforth peringkat ke-2 telah digunakan dalam penyelesaian masalah system dinamik secara tak linear. Dalam kajian ini, bandul Froude telah digunakan. Perbandingan antara kaedah Runge Kutta peringkat ke-4 dengan kaedah Adam Bashforth telah dijalankan untuk menjelaskan keberkesanan antara kedua-dua kaedah dalam pencapaian keputusan yang jitu. Saiz langkah yang berbeza digunakan dalam kertas kerja. Di samping itu, penggunaan bahasa C dalam computer akan dijalankan. Daripada hasil kajian, keputusan daripada kaedah Runge Kutta peringkat ke-4(ERK4) adalah lebih tepat berbanding dengan kaedah Adam Bashforth peringkat ke-2(EAB2).



## ABSTRACT

In this dissertation, the Fourth Order Explicit Runge Kutta method (ERK4) and Second Order Explicit Adam Bashforth method (EAB2) are implemented in solving nonlinear dynamic system such as the Froude pendulum. Comparisons between ERK4 method and EAB2 method will be made to clarify the effectiveness of these two methods to achieve accuracy. Different step sizes are also considered in this dissertation. Besides, a C-language based program will be implemented. From the obtained results, Fourth Order Explicit Runge Kutta Method (ERK4) is found to be more accurate than Second Order Explicit Adam Bashforth Method (EAB2).



## CONTENTS

Page Number

	Page Number
DECLARATION	ii
AUTHENTICATION	iii
ACKNOWLEDGEMENT	iv
ABSTRAK	v
ABSTRACT	vi
CONTENT	vii
LIST OF TABLES	x
LIST OF FIGURES	xi
LIST OF ABBREVIATIONS	xiii
LIST OF SYMBOLS	xiv
<b>CHAPTER 1 INTRODUCTION</b>	<b>1</b>
1.1 Introduction of Ordinary Differential Equation	1
1.2 Nonlinear Dynamic System	3
1.3 Froude Pendulum	4
1.4 Application of Froude Pendulum	5
1.5 Runge-Kutta Method	10
1.6 Multistep Method	12
1.7 Aims And Objective	14
1.8 Scope of Research	14
<b>CHAPTER 2 LITERATURE REVIEW</b>	<b>16</b>
2.1 The Evolution of The Pendulum	16
2.2 Previous Studies On Froude Pendulum	17
2.3 Previous Studies On Nonlinear Dynamic System	18
2.4 Previous Studies On Runge-Kutta Method	19
2.4.1 The Explicit Runge-Kutta Method	19
2.4.2 The Implicit Runge-Kutta Method	20
2.4 Previous Studies On Multistep Method	21
2.5.1 The Explicit Adam-Bashforth Method	22
2.5.2 The Implicit Adam-Moulton Method	23





<b>CHAPTER 3</b>	<b>METHODOLOGY</b>	24
3.1	Introduction	24
3.2	Equation of Motion for A Free Froude Pendulum	26
3.3	Formulation of The Runge-Kutta Methods	30
3.3.1	First-Order Runge-Kutta Method	31
3.3.2	Second-Order Runge-Kutta Method	32
3.3.3	Third-Order Runge-Kutta Method	34
3.3.4	Fourth-Order Runge-Kutta Method	38
3.3.5	High-Order Runge-Kutta Method	44
3.4	Formulation of The Adam-Bashforth Methods	45
3.4.1	First-Order Adam-Bashforth Method	45
3.4.2	Second-Order Adam-Bashforth Method	46
3.4.3	High-Order Adam-Bashforth Method	47
3.5	Absolute Stability	48
3.5.1	Stability Region for Fourth-Order Runge-Kutta Method	49
3.5.2	Stability Region for Second-Order Adam-Bashforth Method	53
3.6	Error Analysis	57
3.6.1	Relative Error	57
3.7	The Coefficient of Determinant, $R^2$	59
3.8	Discretization of The Froude Pendulum	60
3.9	Discretization of ERK4 Method	62
3.9.1	Algorithm for ERK4 Method	63
3.10	Discretization of EAB2 Method	66
3.10.1	Algorithm for EAB2 Method	67
<b>CHAPTER 4</b>	<b>RESULTS</b>	70
4.1	Overview	70
4.2	Numerical Results	71
4.2.1	Comparisons of Errors Versus Number of Iterations at Different Step Sizes	85



<b>CHAPTER 5</b>	<b>DISCUSSIONS AND CONCLUSION</b>	100
5.1	Discussion	100
5.2	Summary And Conclusion	102
5.3	Suggestion for Further Work	105
<b>REFERENCES</b>		106



## LIST OF TABLES

Table No.		Page
3.1	Second order explicit RK scheme in Butcher Tableau	34
3.2	Third order explicit RK scheme in Butcher Tableau	37
3.3	Fourth order explicit RK scheme in Butcher Tableau	43
3.4	Minimum $s$ and number of RKM error coefficients for orders to 8	44
3.5	Coefficients of Adam Bashforth methods up to order 6	47
4.1	Comparisons of theta versus number of iterations at different step sizes with different equations and values of $R^2$	74
4.2	Comparisons of function evaluations, $k$ versus number of iterations at different step sizes with different equations and values of $R^2$	79
4.3	Comparisons of function evaluations, $f$ versus number of iterations at different step sizes with different equations and values of $R^2$	82
4.4	Comparisons of relative error between ERK4 and EAB2 methods at different step sizes with different equations and values of $R^2$	84
4.5	Comparisons of error versus number of iterations at different step sizes with different equations and values of $R^2$	90
4.6	Absolute stability condition for ERK4 method with $h=0.1$ , $h=0.2$ and $h=0.3$	92
4.7	Absolute stability condition for EAB2 method with $h=0.1$ and $h=0.2$ and $h=0.3$	96



## LIST OF FIGURES

Figure No.	Page
1.1 The structure of hand water pump with pendulum	6
1.2 Side view of the pump and the position of the piston, lever and the pendulum during operation of the pump	6
1.3 The structure of pendulum clock	9
3.1 Elements of the methodology	25
3.2 Illustration of a Froude pendulum	26
3.3 Stability plot of the Runge-Kutta method	52
3.4 Stability regions for Adam-Bashforth method	56
3.5 Flowchart of C++ coding for ERK4 method	65
3.6 Flowchart of C++ coding for EAB2 method	69
4.1 Theta versus the number of iterations of ERK4 and EAB2 methods with $h=0.1$ and $w[0]=1$	71
4.2 Theta versus the number of iterations of ERK4 and EAB2 methods with $h=0.2$ and $w[0]=1$	72
4.3 Theta versus the number of iterations of ERK4 and EAB2 methods with $h=0.3$ and $w[0]=1$	72
4.4 Function evaluations, $k$ versus the number of iterations of ERK4 method with $h=0.1$ and $w[0]=1$	75
4.5 Function evaluations, $k$ versus the number of iterations of ERK4 method with $h=0.2$ and $w[0]=1$	76
4.6 Function evaluations, $k$ versus the number of iterations of ERK4 method with $h=0.3$ and $w[0]=1$	76
4.7 Function evaluations, $f$ versus the number of iterations of EAB2 method with $h=0.1$ and $w[0]=1$	80
4.8 Function Evaluation, $f$ versus the number of iterations of EAB2 method with $h=0.2$ and $w[0]=1$	80
4.9 Function Evaluation, $f$ versus the number of iterations of EAB2 method with $h=0.3$ and $w[0]=1$	81



4.10	Relative error versus the different step sizes of each ERK4 and EAB2 methods	83
4.11	Error versus the number of iterations of ERK4 and EAB2 methods with $h=0.1$ and $w[0]=1$	87
4.12	Error versus the number of iterations of ERK4 and EAB2 methods with $h=0.2$ and $w[0]=1$	87
4.13	Error versus the number of iterations of ERK4 and EAB2 methods with $h=0.3$ and $w[0]=1$	88
4.14	Absolute Stability condition for ERK4 method with $h=0.1$	93
4.15	Absolute Stability condition for ERK4 method with $h=0.2$	93
4.16	Absolute Stability condition for ERK4 method with $h=0.3$	94
4.17	Absolute Stability condition for EAB2 method with $h=0.1$	97
4.18	Absolute Stability condition for EAB2 method with $h=0.2$	97
4.19	Absolute Stability condition for EAB2 method with $h=0.3$	98



**LIST OF ABBREVIATIONS**

ODEs	Ordinary Differential Equations
RKM	Runge Kutta -Method
RK4	Runge Kutta-Method of Fourth Order
ERK	Explicit Runge-Kutta
EAB	Explicit Adam-Bashforth
ABM	Adam-Bashforth Method
LMM	Linear Multi Method
ERK4	Explicit-Runge Kutta Method of Fourth Order
EAB2	Explicit Adam-Bashforth Method of Second Order
SSR	Regression
SSE	Error variation
SStotal	Total variation



## LIST OF SYMBOLS

$s$	order of Runge-Kutta method.
$P(s)$	evaluations of $f$ for each timestep
$M(\Omega - \theta)$	frictional torque that related to slipping velocity $\theta$
$m$	mass of pendulum
$I$	total moment of inertia
$l$	distance from the axis of rotation to the center of gravity
$c$	viscous coefficient due to the resistance of the air.
$\Omega$	angular velocity of the rotating shaft.
$w_i$	weighting coefficients
$v$	order of accuracy of the RK method
$c_i$	quadrature nodes
$b_i$	quadrature weights
$a_{ij}$	Runge-Kutta coefficients
$s$	number of stages
$\xi_j$	root of the characteristic polynomial
$P(r)$	Stability polynomial of the RK process
$\frac{d\theta}{d\tau}$	First order derivatives towards a function $\theta$ of $\tau$
$\frac{dw}{dy}$	First order derivatives towards a function $w$ of $y$
$\frac{dy}{dx}$	First order derivatives towards a function $y$ of $x$
$\frac{d^2y}{dx^2}$	Second order derivatives towards a function $y$ of $x$
$\frac{d^2\theta}{d\tau^2}$	Second order derivatives towards a function $\theta$ of $\tau$
$E_{\max}$	Maximum possible absolute error
$R^2$	The coefficient of the determination
$\lambda$	Eigenvalues



$h$	Step size
$r$	Absolute stability condition





## CHAPTER 1

### INTRODUCTION

#### 1.1 Introduction of Ordinary Differential Equation

Differential equation is a type of equation which is expressed in terms of unknown function of several variables that relates the values of the function itself and their derivative. Differential equations commonly have been applied in engineering, physics, economics, and chemistry (Dormand, 1996).

For example, the mathematical models implement the fundamental scientific laws of physics which include principle of conservation of linear momentum, principle of conservation of mass, and principle of conservation of energy in the term of differential equation (Reddy, 2004).

If a solution for a given differential equation is not readily available, the numerical methods are considered to determine the exact form. By using numerical method, this method can compute a solution of accuracy to the differential equation over of period of time as well as solving many practical problems of science and



engineering. A well-known example is simple pendulum which based on the principle of conservation of linear momentum (e.g. Newton's second law of motion) to determine the motion of pendulum (Dormand, 1996; Reddy, 2004)

There are many numerical methods for solving ordinary differential equations (ODEs). These methods are usually categorized as one-step and multistep methods. One of the explicit one-step methods is Runge-Kutta methods. On the other hand, the multistep methods include implicit Adam-Moulton, explicit Adam-Bashforth and Predictor-corrector methods. The approach that used by one-step method is totally different as compared to multistep methods. One-step methods use the solution at a single initial point to get an approximation to the solution of next point. Multistep methods use the sequence of previous solution and derivative values (Dormand, 1996).

There are five types of differential equation; ordinary difference equation, partial difference equation, delay differential equation, stochastic differential equation, and differential algebraic equation. Each of these categories lead to linear and nonlinear system. A differential equation is considered to be linear when the dependent variable and its derivatives occur only to the power one and there are no functions for dependent variable. Otherwise the particular differential equation is considered as nonlinear equation (Wikipedia, 2007a; Dormand, 1996).

Regarding the history of differential equation, Leonhard Euler who also invented Euler's method introduced the theory of differential equations in the year 1768. In the year 1824, Augustin Louis Cauchy used implicit Euler method to prove



convergence of Euler method. John Couch Adams was the first person who designed the multistep method in 1855. After 40 years in 1895, Carle Runge introduced the first Runge-Kutta method. In 1905, Martin Kutta developed the fourth order Runge-Kutta method. During the year of 1910, Lewis Fry Richardson introduced his extrapolation method (Wikipedia, 2007b).

## 1.2 Nonlinear Dynamic System

Dynamic system is the behavior of one or more particles that modelled by differential equation with initial condition. The behaviour of dynamic system such as velocity, position and acceleration. The equation of motion for Froude pendulum is shown as follow: (De Jong, 1991).

$$I \frac{d^2\theta}{dt^2} + (c + M'(\Omega)) \frac{d\theta}{dt} + mgl \sin \theta + \frac{1}{6} M'''(\Omega) \left( \frac{d\theta}{dt} \right)^3 = M(\Omega) \quad (1.1)$$

with initial condition :  $\theta(0) = 0$  .

From the Froude pendulum in equation (1.1), it can be found that it is nonlinear dynamic system because the differential equations involve non-linear terms like  $\sin \theta$  and  $\left( \frac{d\theta}{dt} \right)^3$ . The restoring force of the pendulum that is  $mgl \sin \theta$  where  $\theta$  is the angle of displacement. If the angle of displacement is smaller than  $15^\circ$ , the  $\sin \theta$  is just approximated with simply  $\theta$ . On the contrary, approximation of  $\sin \theta$  is inaccurate and it is considered as nonlinear system if the angle is larger than  $15^\circ$  (Davidson, 2005).



Nonlinear systems seem to provide real natural systems unlike linear systems that make simple equations and behaviour of simple system such as simple pendulum in phase space. Mostly, there are some factors often vary in nonlinear system such as friction forces, damping elements, resistive elements in circuits. It seems that the solution of nonlinear system is very messy if comparing with linear system. The analytic solution to these systems are difficult to be obtained due to the exponential increase error in measurement when the system progresses. As a result, numerical method can only be applied to solve nonlinear system (Davidson, 2005).

### 1.3 Froude Pendulum

Froude's pendulum is a classical dynamic system that generates self-oscillations in a mechanical system with friction. The shaft of the Froude pendulum is connected to an engine that rotates at an angular velocity. It fixed to bearing pivot which swings along the rotating shaft (Dai & Singh, 1998).

Self-excited oscillations are produced due to the friction force between a rotating shaft and the bearing pivot of a Froude pendulum. The increasing amplitude of pendulum depends on damping force and frequency of rotation. This type of pendulum is different from the simple pendulum because of its complex three dimensional motion (Dai & Singh, 1998; Djidjeli *et al.*, 1996).



For simple pendulum, the motion occurred in a two dimension plane such that it does not swing into and out of the page. The actual form of a simple pendulum consists of a long bar suspended at the end of a string that support a small, massive bob. The simple pendulum body suspends from a fixed point so that it can swing under influence of gravity and under its own momentum (Wikipedia, 2008).

#### 1.4 Application of Froude pendulum

##### (i) Hand Water Pump With Pendulum.

Hand water pump with pendulum is a realization of a new and technically original solution for pumping water. The pumping works can be accomplished with less invested energy in the form of pushing of the pendulum. The pendulum generates about 12 times more output energy than the manual input energy required to keep the pendulum swinging. It seems that the pumping work can become easier, long-lasting and effortless if using hand water pump. Hence a user can pump out a 1200 liters of water per hour, without any fatigue and continue with the pumping (Milkovic, 2005).

From Figure 1.1 and 1.2, hand water pump with a pendulum for pumping water consists of a cylinder (1) with a piston (2), lever system (3), a seesaw (4), a pendulum (5), a reservoir (6) and output water pipe (7) (Milkovic, 2005).



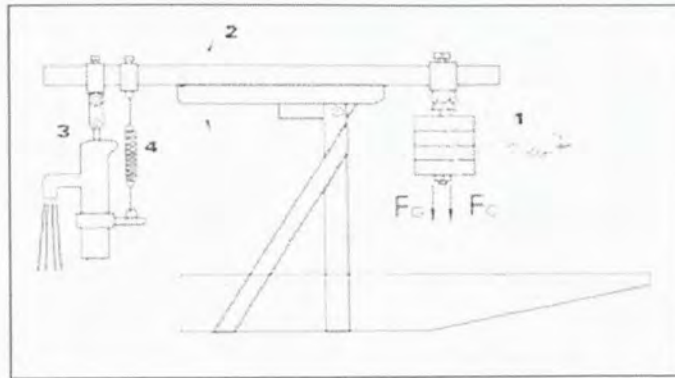


Figure 1.1 The Structure Of Hand Water Pump With Pendulum (Milkovic, 2005).

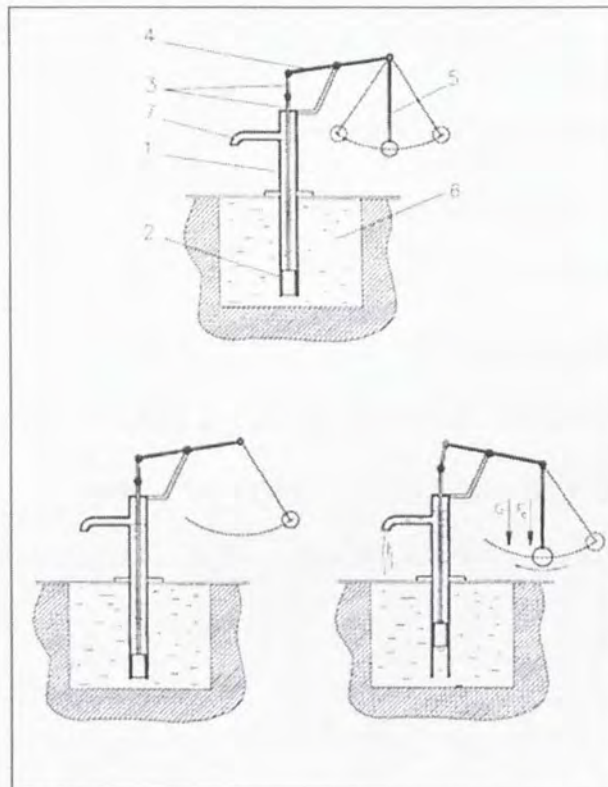


Figure 1.2 Side View Of The Pump And The Position Of The Piston, Lever And The Pendulum During Operation Of The Pump (Milkovic, 2005).

In order to move the piston and lever vertical in the up-down direction, it can give the pendulum which is attached to the end of the lever system a push so that much less effort is needed to pump out water. The action of the hands is required to maintain the amplitude of the oscillation for a continuous intensity of water flow. The pump works well with all sizes of the pendulum, but the best performance is achieved with an amplitude of about  $90^\circ$ . Besides, the see-saw and the piston also oscillating together with the pendulum, the side with the piston should be heavier than the side with the pendulum. It is about 50% heavier than the centrifugal force of the pendulum when its swinging amplitude is  $90^\circ$  (Milkovic, 2005).

To get the water coming out of the output pipe, the pendulum needs to be out of balance and should be occasionally pushed to maintain the amplitude of water. The power of gravitational potential is also used as driving power. With the gravitational potential, the piston will rise to the highest point and the pendulum passes through the bottom vertical position during the swinging. In these moments, the pendulum will overweigh on the see-saw position and in this phase the piston will push the continuous stream of water out of the output pipe (Milkovic, 2005).



The advantages of this new technically hand water pump compared to present hand pump are (Ecosustainable, 2006).

- i) needs the minimum of the effort, because it is only necessary to swing the pendulum and maintain the oscillation for several hours, without any fatigue.
- ii) Both elderly and children can use it because maintaining the oscillation of the pendulum is a easy work and it does not request any special training. So everyone can do it.
- iii) The pendulum can be a children's swing, so that useful work can also be done through their playing.

#### ii) Pendulum Clock.

A pendulum clock is a clock that uses a pendulum and a swinging weight to calculate the accuracy of time period. From its invention in 1656 by Christiaan Huygens until the 1930s, the pendulum clock can be considered as the world's most accurate timekeeper. Pendulum clocks must be stationary to operate; any motion or accelerations will affect the motion of the pendulum, causing inaccuracies, so other mechanisms must be used in portable timekeepers. They are now kept mostly for their decorative and antique value (Ventures, 2004).





## REFERENCES

- Agbeboh, G. U., Aashikpelokhai, U. S. U. & Aigbedion, I. 2007. Implementation of New 4<sup>th</sup> Order Runge Kutta Formula for Solving Initial Value Problem(I.V.Ps). *International Journal Of Physical Sciences* 2 (4), pg. 089-098.
- Allahviranloo, T., Ahmady, N. & Ahmady, E. 2007. Numerical Solution of Fuzzy Differential Equations By Predictor–Corrector Method. *Information Sciences* 177, pg. 1633–1647.
- Ashino, R., Nagase, M., & Vaillancourt., R. 2000. Behind and Beyond the MATLAB ODE Suite. *Computers and Mathematics with Applications* 40, pg. 491-512.
- Atkinson, K. & Han, W. 2004. *Elementary Numerical Analysis*. John Wiley & Sons, Hoboken.
- Attili, B. S., Khalid Furati. & Muhammed I, Syam. 2006. An Efficient Implicit Runge–Kutta Method for Second Order Systems. *Applied Mathematics and Computation* 178, pg. 229–238.
- Bogacki, M. B., Alejski, K. & Szymanowski, J. 1989. The Fast Method of The Solution of A Reacting Distillation problem. *Computers & Chemical Engineering* 13, pg. 1081-1085.
- Brugnano, L. & Trigiante, D. 1995. High-Order Multistep Methods for Boundary Value Problems. *Applied Numerical Mathematics* 18, pg. 79-94.
- Butcher, J. C. 2000. Numerical Methods For Ordinary Differential Equations In The 20<sup>th</sup> Century. *Journal Of Computational And Applied Mathematics* 125, pg. 1-29.
- Butcher, J. C. & Chen, D. J. L. 2000. A New Type of Singly-Implicit Runge–Kutta Method. *Applied Numerical Mathematics* 34, pg. 179–188.



- Butcher, J. C. 2003. *Numerical Methods For Ordinary Differential Equations*. John Wiley & Sons, England.
- Chiou, J. C. & Wu, S. D. 1999. On The Generation of Higher Order Numerical Integration Methods Using Lower Order Adams-Bashforth and Adams-Moulton Methods. *Journal of Computational and Applied Mathematics* **108**, pg. 19-29.
- Dai, L. & Singh, M. C. 1998. Periodic, Quasiperiodic and Chaotic Behaviour of A Driven Froude Pendulum. *Int. J. Non-Linear Mechanics* **33**, pg. 947-965.
- Dai, L. & Singh, M. C. 2003. A New Approach With Piecewise-Constant Arguments To Approximate and Numerical Solutions of Oscillatory Problems. *Journal of Sound And Vibration* **263**, pg. 535-548.
- Dai, L., Xu, L. & Dong, M. 2005. Assessment of The Behaviour of Weakly and Highly Nonlinear Friction-driven Oscillatory System. *Communications In Nonlinear Science And Numerical Simulation* **10**, pg. 57-71.
- Davidson, 2005. *Linear and Non-linear Dynamics* .<http://www.phy.davidson.edu/StuHome/jimn/Chaos/Pages/Theory.html>.
- De Jong, M. L. 1991. *Introduction To Computational Physics*. Addison-Wesley Publishing Company, Reading.
- Davis, P. 1996. *Differential Equation Modeling With MatLab*. Prentice-Hall, Inc, Upper Sadde River, New York.
- Djidjeli, K., Guan, Z., Price, W. G. & Twizell, E. H. 1996. Explicit Finite-Difference Methods for Non-Linear Dynamic System: Froude's Pendulum. *Comput. Methods Appl. Mech. Engrg* **135**, pg. 243-264.



- Dormand, J. R. 1996. *Numerical Methods for Differential Equations*. CRC Press, Boca Raton.
- Ecosustainable, V. 2006. *The Water Hand-Pump With Pendulum*. [http://www.ecosustainablevillage.com/pendulum\\_pump.htm](http://www.ecosustainablevillage.com/pendulum_pump.htm).
- Encyclopedia, 2000. *Pendulum Clock*. <http://www.britannica.com/clockworks/Pendulum.Html>.
- Enns, R. H. & McGuire, G. 1997. *Nonlinear Physics With Maple for Scientists and Engineers*. Boston, Birkhauser.
- Faires, J. D. & Burdan, R. 2003. *Numerical Methods*. Brooks/Cole, Australia.
- Gerisch, A. & Weiner, R. 2003. The Positivity of Low-Order Explicit Runge-Kutta Schemes Applied in Splitting Methods. *Journal Computers and Mathematics with Applications* **45**, pg. 53-67.
- Gingrich, R. & Bennett, T. 2007. *Error Analysis*. <http://www.southernct.edu/~gingrich/erroranalysis2007.pdf>.
- Leveque, R. J. 2007. *Finite Difference Methods for Ordinary and Partial Differential Equations*. <http://www.siam.org/books/ot98/sample/OT98Chapter7.pdf>.
- Lind, D. A., Marchal, W.G. & Wathen, S. A. 2005. *Statistical Techniques In Business and Economi*. McGraw-Hill, Boston.
- Liu, H. Y. & Zhang, K. 2006. Multi-Symplectic Runge-Kutta-Type Methods for Hamiltonian Wave Equation. *Journal Of Numerical Analysis* **26**, pg. 252-271.
- Mathews, J. H. & Fink, K. K. 2004. *Numerical Methods Using MATLAB*. Prentice-Hall, Upper Saddle River.



- Milkovic, V. 2005. *Hand Water Pump With A Pendulum*. <http://www.w3c.org/TR/1999/REC-html>.
- Niesen, J. 2003. *A Priori Estimates for The Global Error Committed By Runge-Kutta Method for A Nonlinear Oscillator*. <http://www.lms.ac.uk/jcm/6/lms2001-024/sub/lms2001-024.pdf>.
- Ostermann, A., Thalhammer, M. & Kirlinger, G. 2004. Stability of Linear Multistep Methods and Applications To Nonlinear Parabolic Problems. *Applied Numerical Mathematics* **48**, pg. 389–407.
- Ramanathan, R. 2002. *Introduction Econometrics With Application*. Michael P. Roche, US.
- Reddy, J. N. 2004. *Nonlinear Finite Element Analysis*. Oxford University Press, New York.
- Rodabaugh, D. J. & Thompson, S. 1979. Low-Order  $A_0$ -Stable Adams-Type Correctors. *Journal of Computational and Applied Mathematics* **5**, pg. 225-233.
- Rodabaugh, D. 1977. Corrector Methods With Increased Ranges of Stability. *Computers & Mathematics with Applications* **3**, pg. 197-201.
- Sauer, T. 2006. *Numerical Analysis*. Pearson Education, Boston.
- Scholarpedia. 2007. *Runge Kutta Methods*. [http://www.scholarpedia.org/article/Runge-Kutta\\_Methods#Implicit\\_Runge\\_Kutta\\_methods](http://www.scholarpedia.org/article/Runge-Kutta_Methods#Implicit_Runge_Kutta_methods).
- Staff, G. A. 2004. *Solving Ordinary And Partial Differential Equations In Time*. [http://heim.ifi.uio.no/~kent\\_and/inf5670/lecture\\_notes10.ps](http://heim.ifi.uio.no/~kent_and/inf5670/lecture_notes10.ps).
- Ventures, A. 2004. *How The Grandfather Clock Works*. <http://www.storesonline.com>.



Wikipedia. 2007a. *Differential Equation*. [http://en.wikipedia.org/wiki/Differential\\_equation](http://en.wikipedia.org/wiki/Differential_equation).

Wikipedia. 2007b. *Numerical Ordinary Differential Equations*. [http://en.wikipedia.org/wiki/Numerical\\_ordinary\\_differential\\_equations](http://en.wikipedia.org/wiki/Numerical_ordinary_differential_equations).

Wikipedia. 2008. *Pendulum*. <http://en.wikipedia.org/wiki/pendulum>.

Wu, X. Y. 2003. A Class of Runge–Kutta Formulae of Order Three and Four With Reduced Evaluations of Function. *Applied Mathematics and Computation* **146**, pg. 417–432.

Zill, D. G. 2005. *A First Course In Differential Equations With Modeling Applications*. Brook, Canada.

