IMPROVING MODIFIED GAUSS-SEIDEL METHOD FOR SOLVING BLACK-SCHOLES PDE USING CRANK-NICOLSON APPROACH

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DECLARATION

I declare that this writing is my own work except for citations and summaries where the sources of every each have been duly acknowledged.

30 April 2008

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ABSTRAK

Dalam kajian ini, persamaan terbitan separa Black-Scholes diselesaikan dalam masalah penilaian Opsyen Jual Eropah. Persamaan terbitan separa tersebut didiskretkan dengan skema tersirat dan Crank-Nicolson. Setelah itu, nilai opsyen dianggarkan dengan menggunakan kaedah Gauss-Seidel Terubahsuai Diperbaiki (IMGS) dan dibandingkan dengan kaedah Gauss-Seidel Terubahsuai (MGS). Kaedah Gauss-Seidel klasik memainkan peranan sebagai kaedah kawalan. Penghampiran demikian diuji untuk saiz grid 512, 1024, 2048, 4096 dan 8192. Melalui eksperimen berangka, kaedah IMGS adalah lebih baik dari segi bilangan lelaran dan masa lelaran. Keputusan adalah lebih jitu apabila kaedah IMGS digunakan menerusi pendekatan Crank-Nicolson.

ABSTRACT

In this study, Black-Scholes PDE is solved in the problem of valuation of European Put Option. The PDE is discretized by implicit and Crank-Nicolson schemes. Then, the option value is approximated by using Improving Modified Gauss-Seidel (IMGS) method and is compared with Modified Gauss-Seidel (MGS) method. The classical Gauss-Seidel (GS) method plays the role as the control method. The approximations are tested with grid sizes of 512, 1024, 2048, 4096 and 8192. By numerical experiments, IMGS method is more superior in terms of number of iteration and computational time. The result is more accurate when IMGS is used with Crank-Nicolson approach.

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LIST OF SYMBOLS

V	value of the option
t	time
S	stock price
σ	volatility
r	risk free interest rate
$\frac{\partial V}{\partial t}$	first derivative towards variable t of the function V
$\frac{\partial V}{\partial S}$	first derivative towards variable S of the function V
$\frac{\partial^2 V}{\partial S^2}$	second derivative towards variable S of the function V
K	strike price
T	expiration time
S(T)	stock price at expiration time
V(S,T)	value of the option at expiration time at certain stock price
I	identity matrix
D	diagonal matrix
U	upper triangular matrix
L	lower triangular matrix
P	preconditioned matrix

LIST OF ABBREVIATIONS

CN	Crank-Nicolson discretization scheme
FDM	Finite Difference Method
GS	Gauss-Seidel iterative method
IMGS	Improving Modified Gauss-Seidel iterative method
MGS	Modified Gauss-Seidel iterative method
PDE	Partial differential equation
SLE	System Of Linear Equation

CHAPTER 1

INTRODUCTION

1.1 Background

Recently, the computational finance has been becoming a field in its own right. In 1995, the first international conference on computational finance was held at Stanford University. Soon after, it was the born of the *Journal of Computational Finance*. These have signified the popularity and the success of computational finance where there was a mass of research and it is ongoing (Tavella, 2002).

1.1.1 Computational Finance

Computational methods are used in solving engineering problems by solving so-called "conservations equations" where analytical solution would not be applicable and numerical solution would be considered. The conservation equations of physics consider relationship between the rates of convection, diffusion, creation and disappearance of mass, momentum and energy. Normally, these relationships are in the form of partial differential equations (PDE). Likewise, financial engineers deal with pricing equations to solve financial instrument which depends on time and the values of other factors. Similarly, these pricing equations are also PDE (Tavella, 2002). However there are a few differences between computation in finance and computation in other fields should be look into.

In numerical methods which normally applied in physical problems in science and engineering, the final experiment result is always derived by considering the initial time condition and the values of other parameters. In contrast, according to Tavella (2002) the pricing equations of pricing instruments are derived by considering the arbitrage and expectations. In other words, in financial computation, the price is always derived by taking into account the final time condition.

In computation in engineering, a very large number of problems can be solved by simply changing the boundary conditions. This relative consensus and stability of the mathematical framework enable it to develop large and flexible software systems to implement certain solution approaches to certain areas of engineering (Tavella, 2002). For examples, mechanical engineers can work on projects ranging from a small machine to a system of operation in a production department, and still use the same methodology, such as finite elements method.

In financial engineering, it is significantly different situation. The pricing of financial instruments is not only repeatedly applying the same numerical methodology with different boundary conditions, it is very particular to a certain financial instruments being considered. In some cases, the pricing equation is unknown. Yet in other cases, the pricing equation is only suitable for certain types of numerical approaches. This implies that the financial engineers must be fluent in a number of computational approaches appropriate for dealing with different instruments (Tavella, 2002).

1.1.2 Types Of Options

Derivative is a financial instrument whose price depends on, or it is derived from, the price of another asset (Almgren, 2002; Brandimarte, 2002; Hull, 2006). The word derivative is not the one in mathematical terms. There are a few types of derivatives such as options, futures, forward contracts, swaps, and the kind traded actively in the over-the-counter market. Option will be the derivative concerned in this dissertation.

Option gives the holder the right to buy or sell a certain underlying asset at a certain price by a certain date from the writer. The right to buy is known as the call option whereas the right to sell is known as the put option. The price and date stated in the option are known as the strike price or exercise price and the expiration date or maturity date respectively (Almgren, 2002; Higham, 2004; Hull, 2006). The underlying assets can be stock, foreign currency, commodities, index, futures, properties and other assets.

The holder of the call option gains profit if the strike price is less than the asset price when the option is exercised. Then the holder can exercise the option with the strike price and sell in the market with the asset price. However, if the strike price is more than the asset price, thus the holder can choose not to exercise the option (Almgren, 2002; Higham, 2004; Hull, 2006).

As for a put option, the holder gains profit when the strike price is more than the asset price in the market during the option is being exercised. Thus, the holder can buy the asset in the market and exercise the option by selling it with the strike price. If the strike price is less than the asset price, the holder can choose not to exercise the option (Almgren, 2002; Higham, 2004; Hull, 2006).

However, the holder has to pay an amount of money to the writer of the option to purchase the option. Hence, this amount of money is the price or the value of the option. The following are the types of options where they have nothing to do with the geographical location:

a. European Option

European option is the option that can only be exercised on maturity. A European call option gives its holder the right to buy from the writer a prescribed asset with the stated exercise price at the stated maturity date. On the other way round, a European put option gives its holder the right to sell to the writer a prescribed asset with the stated strike price at the expiration date (Higham, 2004; Hull, 2006).

b. American Option

American option is the option that can be exercised at any time up to the maturity. An American option is more widely traded than the European option. An American call option gives its holder the right to buy from the writer an underlying asset with the strike price at any time between the purchased date of the option and the maturity date.

As for an American put option, it gives the holder the right to sell to the writer an underlying asset with the strike price at any time between the purchased date of the option and the expiration date in future (Higham, 2004; Hull, 2006).

c. Exotic Option

European and American options are known as plain vanilla products. Exotic options are the nonstandard products that have been created by financial engineers. Exotic products are developed for a variety of reasons in order to meet the market need. Examples of exotic options are Nonstandard American option, Forward start option, Compound option, Chooser option, Barrier option, Binary option, Lookback option, Shout option, Asian option, Russian option and many more (Almgren, 2002; Higham, 2004; Hull, 2006).

1.1.3 Stock Option

The underlying asset that is interested in this dissertation is the stock. A contract of the stock option gives the holder the right to buy or sell 100 shares at the stated strike price as the shares themselves are normally traded in lots of 100. The factor that affecting the stock option price are the current stock price, strike price, time of expiration, volatility of the stock price, the risk-free interest rate, and the dividends expected during the life of the option (Hull, 2006).

1.1.4 Problem Formulation

The paper of Fischer Black and Myron S. Scholes in 1973, *The Pricing of Options and Corporate Liabilities,* derives the key equation of Robert C. Merton in stock option pricing and developed the well known Black-Scholes PDE or also known as Black-Scholes-Merton PDE (Hull, 2006). For centuries, options have been traded informally until as late as 1973 when the first trading of options at an exchange, which was the Chicago Board Options Exchange (CBOE). Hence, there was a happy coincidence between the arrival of the quantitative understanding of options and the development of market institutions to trade in options. In 1997, their work was recognized where Robert C. Merton and Myron Scholes were awarded the Nobel Prize in Economics. If their partner, Fischer Black, had been alive, he would have shared the price (Shah,1997) .

The Black-Scholes PDE is as follows (Almgren, 2002; Black & Scholes, 1973; Brandimarte, 2002; Goto *et al.*, 2007; Higham, 2004; Hull, 2006; Shah, 1997; Tavella, 2002; Zhao *et al.*, 2007):

$$\frac{\partial V}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} + rV \tag{1.1}$$

where,

V is the value of the options,

t is the time,

S is the stock price,

σ is the volatility of a stock,

r is the risk free interest rate.

The Black-Scholes PDE is subjected to the following final time condition (Brandimarte, 2002; Goto *et al.*, 2007; Higham, 2004; Hull, 2006; Tavella, 2002) :

For European call option:

$$V(T,S) = \max(0, S(T) - K) \tag{1.2}$$

For European put option:

$$V(T,S) = \max(0, K - S(T)) \tag{1.3}$$

where,

K is the strike price,

T is the expiration time,

$S(T)$ is the stock price at expiration time

$V(T,S)$ is the value of the option at expiration time at certain stock price.

Refer to equation (1.2) regarding the European call option, at the expiration time, *T*, if $S(T) > K$, then the holder may exercise the option by purchasing the asset with K and sell it in the market for $S(T)$ to gain an amount $S(T) - K$. On the other hand, if $S(T) \leq K$, then the holder gains nothing (Higham, 2004; Hull, 2006).

Next consider equation (1.3) regarding the European put option, at the expiration time, T when $S(T) < K$, thus the holder may buy the asset at $S(T)$ in the market and exercise the option by selling it at K obtaining a profit of $K - S(T)$. On the other hand, if $S(T) \geq K$, then the holder should do nothing (Higham, 2004; Hull, 2006).

According to Black and Scholes (1973), the author of the Black-Scholes PDE, the PDE is assuming an "ideal conditions":

i. The risk free interest rate is known and is constant through time.

ii. The stock pays no dividends or other distributions.

iii. The stock price is random in continuous time with a variance rate proportional to the square of the stock price. Therefore, the distribution of possible stock price at the end of any finite interval is lognormal. The variance rate of return on the stock is constant.

iv. There are no transactions costs or taxes in trading the stock or option.

v. It is able to borrow any fraction of the price of a security to buy it or hold it, at the short term interest rate.

vi. There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

vii. It is a European option which means can only be exercised at maturity.

Those were the assumptions given by Black and Scholes (1973), but the assumption is improved today and can be applied to other option not only the European option.

1.2 Numerical Methods

As analytical solutions like Black-Scholes formula are not available in general, one must often apply the numerical methods (Brandimarte, 2002). Besides that, analytical solution is not practical to solve a larger variety of stock price in continuous time. Hence, approximated solution which is near to the exact solution, simple and faster have to be applied to solve the problems. Figure 1.1 shows the classification of numerical method family used in solving differential equation.

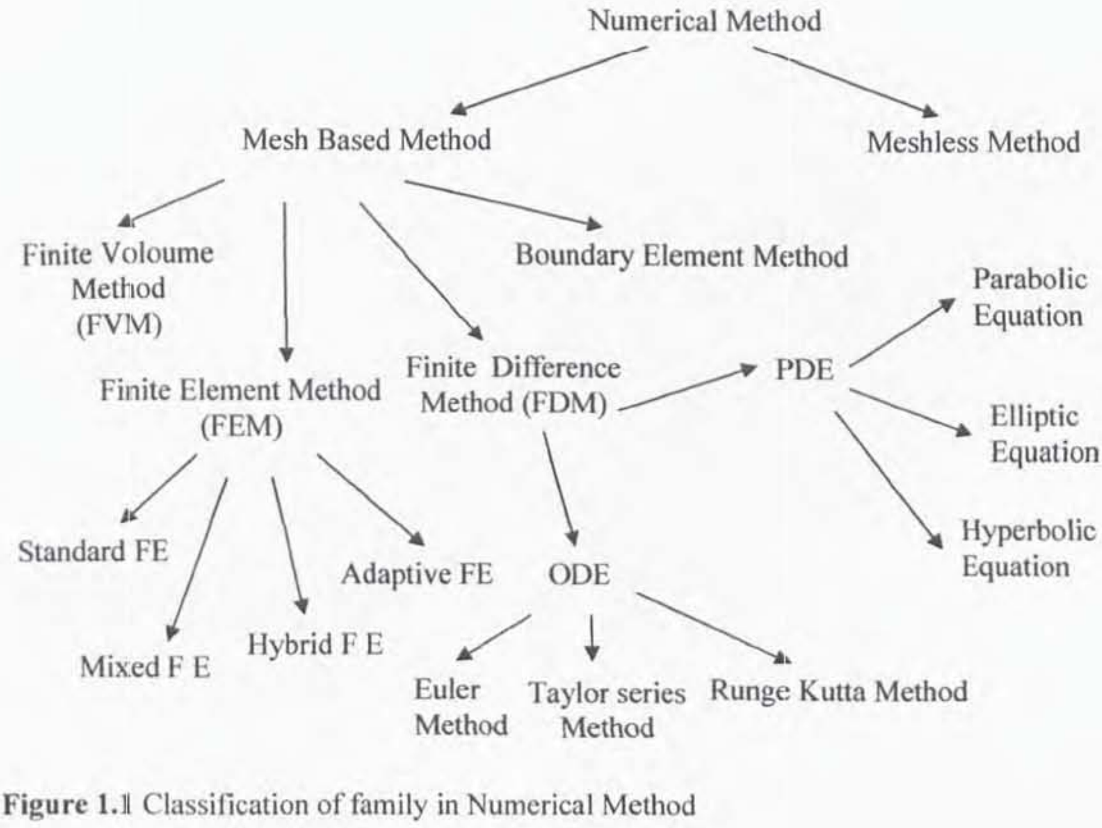

Figure 1.1 Classification of family in Numerical Method

1.2.1 Finite Difference Method

As shown in Figure 1.1, there are numerous approaches can be used in numerical methods. FDM is the most common method used by most researchers.

To obtain the approximate solution for a PDE, the first step is to discretize (Higham, 2004). For a parabolic equation like the Black-Scholes PDE, we discretize it by using Explicit, Implicit and Crank-Nicolson (CN) scheme which are the schemes in FDM. In addition, Theta Scheme is the general scheme for the three schemes above. These schemes are using finite difference operator which are manipulated by the Taylor series expansion which is shown in equation (1.4) where u(x) as the function:

$$u(x+h) = u(x) + \frac{h}{1!}u'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \cdots \tag{1.4}$$

Table 1.1 Common finite difference operators

Operator	Definition
Forward Difference	$\frac{u_{k+1} - u_k}{h}$
Backward Difference	$\frac{u_k - u_{k-1}}{h}$
Central Difference	$\frac{u_{k+1} - u_{k-1}}{2h}$
Second order central difference	$\frac{u_{k+1} - 2u_k + u_{k-1}}{h}$

Let say we consider a simple heat equation such as equation (1.5):

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{1.5}$$

Then the equation (1.5) above can be represented by the Theta Scheme as shown below:

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta \tau} = \alpha \left(\frac{(1-\theta)\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) + \theta\left(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right)}{\Delta x^2} \right) \tag{1.6}$$

Thus, by using equation (1.6), explicit, implicit and CN schemes can be easily formulated by giving the following value:

i. $\theta = 0$ will be explicit scheme.

ii. $\theta = 1$ will be implicit scheme.

iii. $\theta = \frac{1}{2}$ will be CN scheme.

Explicit scheme is so-called explicit because it can be solved directly after discretization process. Yet, implicit and CN schemes have to be solved implicitly by using a solver to solve the system of linear equation generated.

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