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## DECLARATION

I hereby declare that this dissertation contains my original research work. Source of findings reviewed herein have been duly acknowledged.

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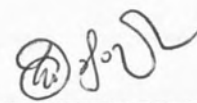
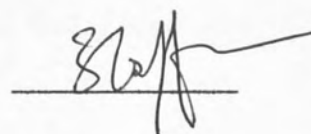
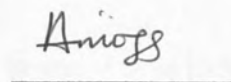
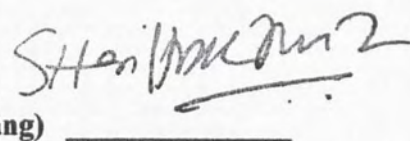


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## ABSTRACT

This dissertation was intended to offer an explanation of the Platonic Solids, Archimedean Solids and Kepler Poincot Solids, at once simple and practical but not too speculative. There are some characteristic and properties have been observed and discussed. Every polyhedron follows the Euler's Formula and has its dual polyhedron. However, there is an exception of Euler's Formula for Kepler Poincot Solids. All the Platonic Solids, Archimedean Solids and Kepler Poincot solids can be described in an easier form like schlaflli symbol and vertices configuration. The tessellation of polyhedron shows its arrangement of polygonal faces. Archimedean can be formed by truncation and snubbing process of Platonic Solids. On the other hand, Kepler Poincot can be constructed by stellations of Platonic Solids. Some regular polyhedra share the common vertex arrangement or same edge arrangement.



## ABSTRAK

Kajian ini menyelidik tentang Pepejal Platonik, Pepejal Archimedean, dan Pepejal Kepler Poincaré. Ciri-ciri am telah dikaji dalam kajian ini. Setiap polihedron mematuhi hukum Euler dan mempunyai polyhedron dual. Namun begitu, terdapat pengecualian untuk kes Pepejal Kepler Poincaré dalam hukum Euler. Semua polihedron seragam dan semiseragam dibentang dalam bentuk yang lebih mudah iaitu simbol Schlafli dan konfigurasi puncak. *Tessellation* bagi polyhedron menggambarkan susunan untuk suatu polyhedron. Pepejal Archimedean boleh dihasilkan dengan kaedah *truncation* dan *snubbing*. Di samping itu, Pepejal Kepler Poincaré boleh dibentuk melalui *stellation* bagi Pepejal Platonik. Beberapa polyhedron seragam menikmati susunan puncak yang sama atau susunan tepi yang sama.



## CONTENTS

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	Page Number
DECLARATION	ii
CERTIFICATION	iii
ACKNOWLEDGEMENT	iv
ABSTRCT	v
ABSTRAK	vi
LIST OF CONTENTS	vii
LIST OF TABLES	x
LIST OF FIGURES	xi
TABLE OF SYMBOLS	xiv
<b>CHAPTER 1 INTRODUCTION</b>	<b>1</b>
1.1 What Is Polyhedron	1
1.1.1 Platonic Solids	5
1.1.2 Archimedean Solids	6
1.1.3 Kepler Poinot Solids	7
1.2 Historical Background	8
1.2.1 Platonic Solids	9
1.2.2 Archimedean Solids	10
1.2.3 Kepler Poinot Solids	11
1.3 Why Study Polyhedra?	11
1.4 Objective of Study	12
1.5 Scope of Study	13
<b>CHAPTER 2 LITERATURE REVIEW</b>	<b>14</b>
2.1 Overview	14
2.2 Previous Research on Polyhedra	15
2.3 Graph as Polyhedra; Polyhedra as Graph	17
2.4 Angles of Polyhedra	17
2.5 Volume of Polyhedra	18
<b>CHAPTER 3 METHODOLOGY</b>	<b>20</b>
3.1 Euler's Formula	20
3.1.1 Euler's own Proof	22





3.1.2	Legendre's Proof	23
3.1.3	Cauchy's Proof	25
3.1.4	Comparison among Euler's Proof, Legendre's Proof and Cauchy's Proof	27
3.1.5	What did Euler's Formula tell us?	27
3.2	Formula Derivations for Polyhedra	28
3.2.1	Dihedral Angle	28
3.2.2	Apothem, polygon area and surface area	30
3.2.3	Inradius and circumradius	31
3.2.4	Volume	32
3.3	Definition of Tessellation, Schläfli Symbol and Vertices Configuration	32
3.4	There Are Only Five Platonic Solids	33
3.4.1	Proved by Descartes Theorem	34
3.4.2	Proved by Sum of Interior Angles	36
<b>CHAPTER 4 RESULT AND DISCUSSION</b>		39
4.1	Number of Vertices, Edges, Faces and Euler's Formula	39
4.1.1	Platonic Solids	39
4.1.2	Archimedean Solids	40
4.1.3	Kepler Poincot Solids	42
4.2	General formula of Angles, Radius and Volume for Platonic Solids	44
4.2.1	Tetrahedron	45
4.2.2	Cube	46
4.2.3	Octahedron	48
4.2.4	Dodecahedron	49
4.2.5	Icosahedron	50
4.3	Tessellation, Schläfli Symbol, and Vertices configuration	52
4.3.1	Platonic Solids	52
4.3.2	Archimedean Solids	53
4.3.3	Kepler Pointsot Solids	57
4.4	Maple Output of Polyhedra	58
4.4.1	Platonic Solids	58



4.4.2	Archimedean Solids	60
4.4.3	Kepler Poincot Solids	65
4.5	Duality	66
4.5.1	Platonic Solids	67
4.5.2	Archimedean Solids	69
4.5.3	Kepler Poincot Solids	74
4.6	Relationship among Polyhedra	77
4.6.1	Relationships between Platonic Solids	77
4.6.2	Relationship between Platonic Solids and Archimedean Solids	78
4.6.3	Relationship between regular polyhedra	78
<b>CHAPTER 5 CONCLUSION AND SUGGESTION</b>		
5.1	Conclusion	80
5.2	Suggestion on further research	83
	REFERENCES	85
	APPENDIX	87
	GLOSSARY	103



## LIST OF TABLES

Table Number		Page Number
1.1	Naming polyhedra	3
3.1	Differences for Euler's, Legendre's and Cauchy's proof	27
4.1	Summary table of number of vertices, edges and faces for Platonic Solids	40
4.2	Summary table of number of vertices, edges and faces for Archimedean Solids	41
4.3	Summary table of number of vertices, edges and faces for Kepler Poinot Solids	43
4.4	Summary table for general formula	44
4.5	Tessellation, schlafl symbol and vertices configurations for Platonic Solids	52
4.6	Tessellation, schlafl symbol and vertices configurations for Archimedean Solids	54
4.7	Schlafl symbol for Kepler Poinot Solids	57
4.8	Duals of polyhdra	75
4.9	The relationship between regular polyhedra	79



## LIST OF FIGURES

Figure Number	Page Number
1.1 Basic terms of polyhedron	1
1.2 Dihedral angle	2
1.3 Solid angle	2
1.4 Vertex figure	5
1.5 Platonic Solids	6
1.6 Archimedean Solids	7
1.7 Kepler Poincot Solids	8
3.1 Octahedron before truncation process	23
3.2 Octahedron after truncation process	23
3.3 Radial projection of octahedron	24
3.4 Planar network of cube	26
3.5 Planar network of cube after adding diagonals to any non-triangular faces	26
3.6 Planar network of cube after removing of one triangular Face	26
3.7 Example of three triangles forming a vertex	28
3.8 Diagram of n-gon	30
3.9 Diagram to show the inradius and circumradius	31
3.10 Tetrahedron	36
3.11 Octahedron	37
3.12 Icosahedron	37
3.13 Cube	37
3.14 Dodecahedron	38
4.1 Maple output of tetrahedron	58
4.2 Maple output of cube	59
4.3 Maple output of octahedron	59
4.4 Maple output of dodecahedron	59
4.5 Maple output of icosahedron	60
4.6 Maple output of truncated tetrahedron	60



4.7	Maple output of truncated cube	61
4.8	Maple output of truncated octahedron	61
4.9	Maple output of truncated dodecahedron	61
4.10	Maple output of truncated icosahedron	62
4.11	Maple output of cuboctahedron	62
4.12	Maple output of icosidodecahedron	62
4.13	Maple output of rhombicuboctahedron	63
4.14	Maple output of rhombicosidodecahedron	63
4.15	Maple output of great rhombicuboctahedron	63
4.16	Maple output of great rhombicosidodecahedron	64
4.17	Maple output of snub cube	64
4.18	Maple output of snub dodecahedron	64
4.19	Maple output of small stellated dodecahedron	65
4.20	Maple output of great stellated dodecahedron	65
4.21	Maple output of great icosahedron	66
4.22	Maple output of great dodecahedron	66
4.23	Maple output for dual of tetrahedron	67
4.24	Maple output for dual of cube	68
4.25	Maple output for dual of octahedron	68
4.26	Maple output or dual of dodecahedron	68
4.27	Maple output for dual of icosahedron	69
4.28	Maple output for dual of truncated tetrahedron	69
4.29	Maple output for dual of truncated cube	70
4.30	Maple output for dual of truncated octahedron	70
4.31	Maple output for dual of truncated octahedron	70
4.32	Maple output for dual of truncated icosahedron	71
4.33	Maple output for dual of cuboctahedron	71
4.34	Maple output for dual of icosidodecahedron	71
4.35	Maple output for dual of rhombicuboctahedron	72
4.36	Maple output for dual of rhombicosidodecahedron	72
4.37	Maple output for dual of great rhombicuboctahedron	72
4.38	Maple output for dual of great rhombicosidodecahedron	73
4.39	Maple output for dual of snub cube	73



4.40	Maple output for dual of snub dodecahedron	73
4.41	Maple output for dual of small stellated dodecahedron	74
4.42	Maple output for dual of great stellated dodecahedron	74
4.43	Maple output for dual of great icosahedron	75
4.44	Maple output for dual of great dodecahedron	75
4.45	Interrelationship of tetrahedron, cube and octahedron	77



**TABLE OF SYMBOLS**

=	equal
+	addition
-	substraction
$\Sigma$	summation



## CHAPTER 1

### INTRODUCTION

#### 1.1 What Is Polyhedron?

A polyhedron is a closed, three-dimensional version of polygon. It is composed of polygons connected at their edge in order to enclose space. In another word, it is a geometric object with flat surfaces and straight edges. The plural of polyhedron is polyhedra or polyhedrons (Alexander, 2003).

The word polyhedron is derived from Greek in which poly- means “many” and -edron means “base”, “seat” or “face”. A polyhedron consists of vertex (vertices in plural), edges and faces as shown in Figure 1.1 below.

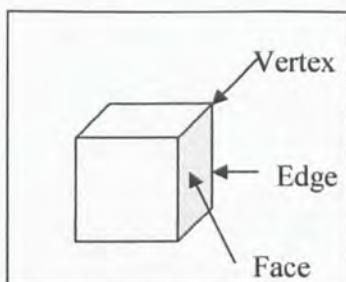


Figure 1.1 Basic terms of polyhedron



A face of the polyhedron is each of its polygonal sides whereas an edge is a line segment where two faces meet, it is formed by the sides of two faces. A point called vertex is formed when few edges and faces are meet. This modified definition is given by Cromwell (1997) in his book “Polyhedra”.

According to Henderson & Daina (1996) in “Experiencing Geometry Euclidean and Non-Euclidean with History”, there are few types of angles involved in polyhedron. Plane angle is the angle at the corner of a polygonal face. Dihedral angle is the angle created by two adjacent faces which have two sides joined to form an edge; it can be measured by the angle of two rays that lie in the planes which have a vertex on the edge and perpendicular to the edge as shown in Figure 1.2. On the other hand, solid angle is the portion of interior of a polyhedron at a vertex which is surrounded by three or more plane angles as shown in Figure 1.3 below.

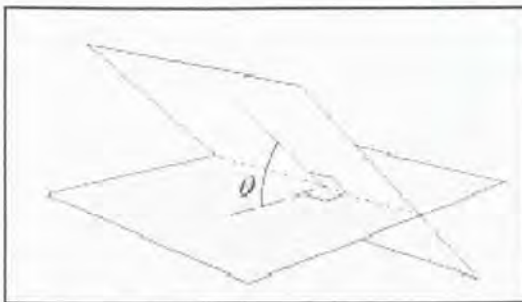


Figure 1.2 Dihedral angle

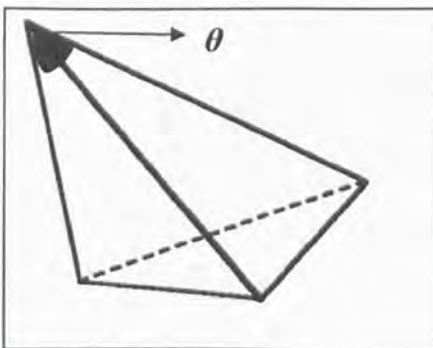


Figure 1.3 Solid angle

There are few common properties shared by polyhedra.

i. Naming of polyhedra

Naming of polyhedra is almost similar to the naming of polygon but more complicated. Naming of polyhedra based on Classical Greek according to its number of faces as shown in Table 1.1 below. Though, there are some special polyhedra have their own name such as Szilassi polyhedron.

Table 1.1 Naming polyhedra

Number of faces	Polyhedron
4	Tetrahedron
5	Pentahedron
6	Hexahedron
7	Heptahedron
8	Octahedron
9	Nonahedron
10	Decahedron
11	Undecahedron
12	Dodecahedron
14	Tetradecahedron
20	Icosahedron
24	Icositetrahedron
30	Tricontahedron



Table 1.1 Naming polyhedra (continued)

32	Icosidodecahedron
60	Hexecontahedron
90	enneacontahedron

## ii. Edges

Edges have two important features that it just joins two faces and two vertices with the exception of those complex polyhedra which constructed in unitary three-space with six dimensions.

## iii. Euler Characteristic

All the polyhedron obey Euler Characteristic with  $V-E+F= 2$ , where  $V$  is the number of vertices,  $E$  is the number of edges, and  $F$  represents the number of faces. According to Malkevitch, this formula was discovered in around 1750 by Euler, and first proven by Legendre in 1794.

## iv. Duality

There exists a dual polyhedron for every polyhedron. Dual polyhedron is the polyhedron that associates with another polyhedron with one faces correspond to the vertices of another and vice versa. Thus, the number of faces of one polyhedron is equal to the number of vertices of its dual polyhedron.



#### v. Vertex figure

A vertex figure which consisting of the vertices which join to it can be defined from every vertex. In another word, vertex figure is formed when a polyhedron is truncated.

Vertex figure is illustrated in Figure 1.4 below.

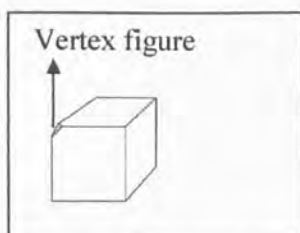


Figure 1.4 Vertex figure

As indicated by Alexandrov (2005), a polyhedron is said to be convex if it composed of many planar polygons so that it is possible for a polygon to pass from one to another by polygons which having common sides of segment of sides and the whole figure lies on one side of the plane of each constituent polygons.

A polyhedron is said to be regular if its faces are made up of regular polygons that have equal sides and angles. It means that every face consists of same number of vertices and vice versa.

#### 1.1.1 Platonic Solids

Another name for platonic solids is regular polyhedron. It is convex regular polyhedra. Platonic solids are perfectly regular solids with congruent faces which made up of

identical polygons. It follows few of conditions where all its faces are identical, all its dihedral angles are the same and all its sides are equal.

There are only five platonic solids exist. The five platonic solids are tetrahedron, cube, octahedron, dodecahedron, and icosahedron as shown in Figure 1.5 below.

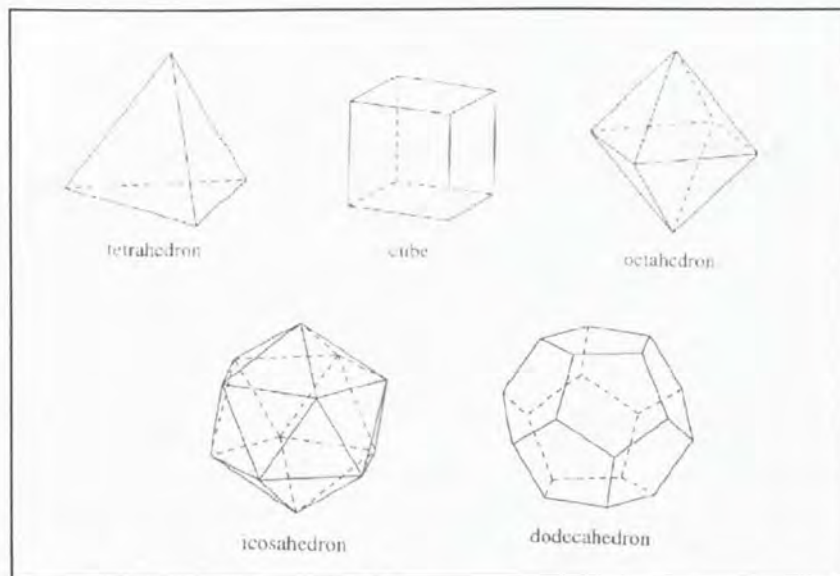


Figure 1.5 Platonic Solids

### 1.1.2 Archimedean Solids

Sometimes are called Archimedean Polyhedra which are convex semi regular polyhedra. Archimedean polyhedra are composed of equilateral and equiangular but not similar polygons. Its faces might be made up of different types of polygons but its vertices are identical. Every vertex of Archimedean Solids is congruent which means the faces must be arranged in the same order around each vertex.

There are thirteen Archimedean Solids which named truncated tetrahedron, truncated cube, truncated octahedron, truncated dodecahedron, truncated icosahedron, cuboctahedron, icosidodecahedron, rhombicuboctahedron, rhombicosidodecahedron, great rhombicuboctahedron, great rhombicosidodecahedron, snub cube, and snub dodecahedron. All these Archimedean solids are shown in the Figure 1.6 below.

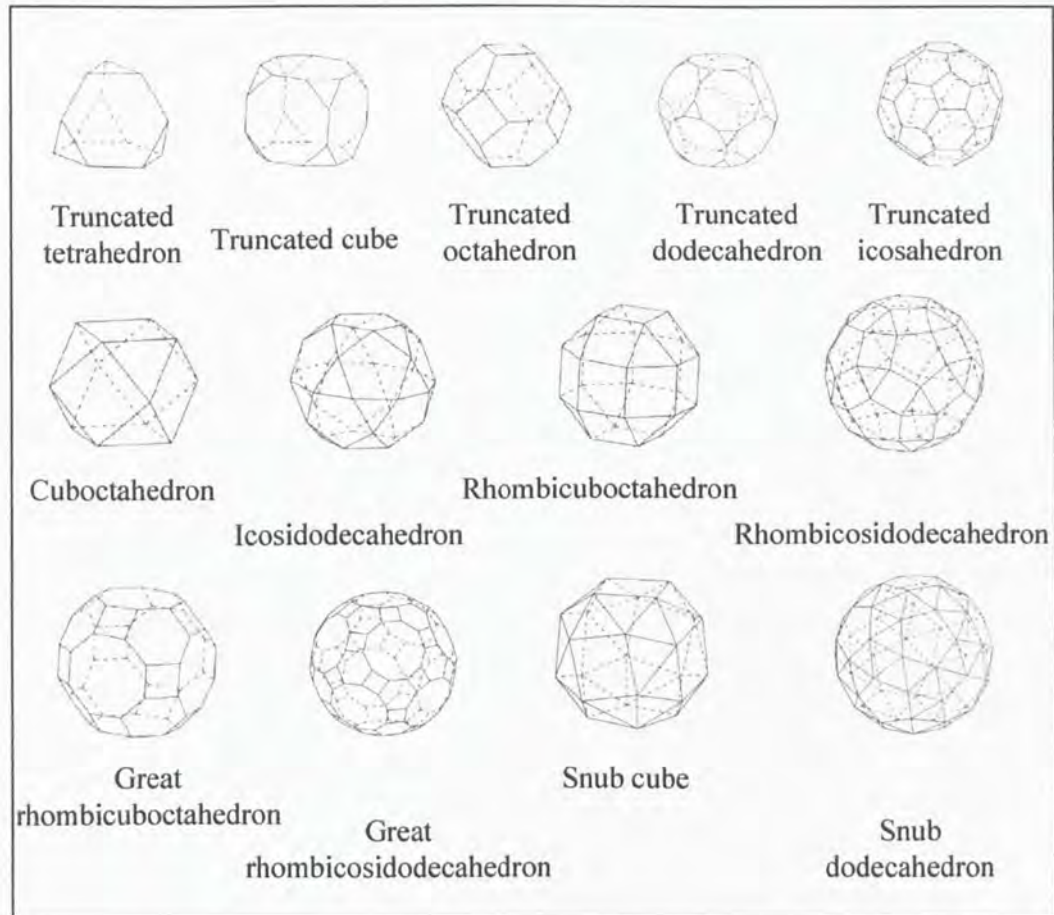


Figure 1.6 Archimedean Solids

### 1.1.3 Kepler Poinot Solids

There exist four types of regular non-convex polyhedra or can be called regular concave polyhedra which known as Kepler Poinot Solids. They are polyhedra that

made up of regular concave polygons with intersecting planes. The number of faces meeting at each vertex is the same.

Small stellated dodecahedron and great stellated dodecahedron are known as Kepler solids which discovered by astronomer Johannes Kepler whereas the great icosahedron and great dodecahedron discovered by French mathematician, Louis Poinso, named as Poinso solids. The four Kepler Poinso solids are shown in the Figure 1.7 below.

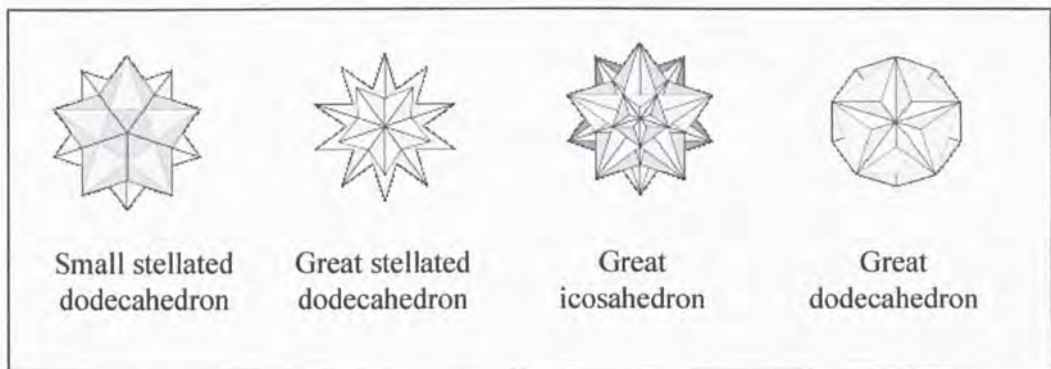


Figure 1.7 Kepler Poinso Solids

## 1.2 Historical Background

Carved stones from the Neolithic time about 1000 years before Plato have been discovered in Scotland. They are oldest known regular polyhedra. Several of the regular polyhedra and the semiregular cuboctahedron were known in Babylon, Egypt, India and China. For example, the ancient pyramids at Giza, Egypt was built over 4500 years ago. The geometry was developed during the golden period of ancient Greek culture.

The earliest written record of these polyhedra shapes are came from classical Greek. After that, Islamic scholars continued to make it advances, for example in the tenth century Abu'l Wafa describes two and three dimensional construction. Meanwhile in China, the formula for the volume of a truncated pyramid was derived by Liu Hui by dissecting pyramid into a central square-based prism.

In the Ranaissance, the Greek literature was translated to Latin from Arabic and later from Greek. Pierro della Francesca (1410-1492) studied the five regular platonic solids and six of the semi regular Archimedean polyhedra. Leon Battista Alberti (1404-1472) and Pierro della Francesca introduced the theory of the perspective according to Vitruvius (50 B.C).

Luca Pacioli (1445-1517) described the Platonic solids and six of Archimedes polyhedra. On the other hand Leonardo da Vinci drew the polyhedra in Pacioli's book "De Divina Proportione" but he did not know the collection by Pappus and thought that there were an unlimited number of semi regular polyhedra. Albrecht Dürer designed the polygons of the regular polyhedra and nine of the semiregular polyhedra onto a plane in the form of "nets" (flat patterns) in Underweisung year 1525. (Cromwell, 1997)

### 1.2.1 Platonic Solids

The regular polyhedra were attributed by ancient Greek and studied by a group of Greek mathematicians under the supervision of Pythagoreans as far back as 500BC.





## REFERENCES

- Alexandrov, A.D. 2005. *Convex Polyhedra*. Springer Berlin Heidelberg, New York.
- Alexander, D.C. & Koerberlein, G.M. 2003. *Elementary Geometry for College Students*. Houghton Mifflin Company, New York.
- Chong, W.H. 2002. *Polyhedra*. Undergraduate Research Opportunity Programme in Science. National University of Singapore, Singapore (Tidak diterbitkan).
- Coxeter, H.C.M. 1973. *Regular Polytopes*. Ed. Ke-3. Dover Publication, Inc, New York. 4-73.
- Cromwell, P.R. 1997. *Polyhedra*. Cambridge University Press, Cambridge.
- Dobrovolskis, A.R. 1996. Inertia of any Polyhedra. *Icarus* **124** (2): 698-704.
- Grunbaum, B. 2007. Graphs of polyhedra; polyhedra as graph. *Discrete Mathematics* **307** (3-5): 445-463.
- Henderson, D.W. & Daina. T. 1996. *Experiencing Geometry Euclidean and Non-Euclidean with History*. Ed. ke-3. Pearson Prentice Hall, New York.
- Jacobs, H.R. 2004. *Geometry: Seeing, Doing, Understanding*. Ed. ke-3. W.H.Freeman and Company, New York.
- Kavitha, K. 2002. *Polyhedra*. Undergraduate Research Opportunity Programme in Science. National University of Singapore, Singapore (Tidak diterbitkan).
- Malkevitch, J. 2007. Euler's polyhedral formula. *Feature Column Monthly Essays on Mathematical Topics*. Retrieved 20 August 2007 from [www.ams.org/featurecolumn/archieve/eulers-formula.html](http://www.ams.org/featurecolumn/archieve/eulers-formula.html).



- Ong, H.L., Huang, H.C. & Huin, W.M. 2003. Finding the exact volume of a polyhedron. *Advances in Engineering Software* **34** (6): 351-356.
- Pook,L. 2003. *Flexagons Inside Out*. Cambridge University Press, Cambridge.
- Rovenski,V. 2000. *Geometry of Curve and Surfaces with MAPLE*. Birkhauser Boston, New York.
- Stahl, S. 2003. *Geometry: From Euclid to Knots*. Pearson Education, New Jersey.
- Wenninger,M.J. 1983. *Dual Models*. Cambridge University Press, Cambridge.
- Yip, B. & Klette, R. 2003. Angle counts for isothetic polygons and polyhedra. *Pattern Recognition Letters* **24** (9-10): 1275-1278.

