TOWER OF HANOI: VISUALISATION OF THE BI-COLOUR PENTAD TOWER IN VRML

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DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF BACHELOR OF SCIENCE WITH HONOURS

MATHEMATICS WITH COMPUTER GRAPHICS PROGRAMME
SCHOOL OF SCIENCE AND TECHNOLOGY
UNIVERSITI MALAYSIA SABAH

April 2007

PUMS 99:1

UNIVERSITI MALAYSIA SABAH

BORANG PENGESAHAN STATUS TESIS@

JUDUL: VISUALISATION OF THE BI-COLOUR PENTAD TOWER IN VRML.

Ijazah: DEGREE OF BACHELOR OF SCIENCE WITH HONOURS.

SESI PENGAJIAN: 2004/2005

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DECLARATION

I hereby declare that this dissertation contains my original research work. Sources of findings reviewed herein have been duly acknowledged.

12 March 2007

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ACKNOWLEDGEMENT

I owe the completion of this project to my advisor, Mr. Tiong Kung Ming, who painstakingly read each draft and gave me confidence in my ideas. He gave me invaluable advice on how to conduct a research project because it was so easy to over estimate one's capabilities and under estimate the difficulty of a problem and the time to work it out in detail. Mr. Victor is also the expert on games in my research.

A special thanks goes to my colleagues who have helped me in many ways. I had many hours of valuable discussion with Ms. Navaneesha and Ms. Sherina Vijaya who were one of my sources of inspirations. Not forgetting to thank Mr. Steven Wong who helped with my VRML programming. To Mr. Jassimran Singh, Ms. Rubini Krishnan and Ms. Shasha Chako, I owe their moral support.

Lastly, I must thank my family for tolerating me and standing behind me when I was struggling through uninspired periods. My mother gave me the confidence to carry on and showed me a new meaning to the word patience.

ABSTRACT

The Bi-Colour Pentad Tower is a variation from the Tower of Hanoi game. The Bi-Colour Pentad Tower consists of five pegs, ($N = 5$) and n discs. There are two different colours of discs. The main aim of this game is to form two towers that consist of two different colours, in this case, to form a blue and red tower of discs on different pegs. The number of minimal moves was studied and a general formula was generated. The formula for $1 \leq n \leq 3$ discs is $2(n-1)$, where $n = 2 \; and \; 3$. When $n = 1$, only one move is required. The Hanoi Graphs for the Bi-Colour Pentad Tower is drawn for one, two and three discs. The Hanoi Graphs shows all the possible movement of the discs, including the fastest and the longest move to solve the game. The Hanoi Graphs for each number of discs are drawn using Virtual Reality Modelling Language (VRML) in a 3D view. The Hanoi Graphs for the Bi-Colour Pentad Tower forms a pyramid shape. As the number of discs increases, the Hanoi Graphs becomes more complicated. The Hanoi Graphs shows a repetition pattern that consists of pyramids combination. To compare the Hanoi Graphs for the Bi-Colour Pentad Tower and other modifications of the Hanoi Tower game, the Hanoi Graphs for four pegs ($N = 4$) is drawn using VRML for one, two and three discs. The same recursive pattern of tetrahedrons and pyramids is seen for both $N = 4$ and $N = 5$ pegs of the Hanoi Game.

VISUALISASI MENARA HANOI DWI-WARNA DENGAN VRML

ABSTRAK

Menara Hanoi Dwi-Warna adalah satu modifikasi bagi permainan Menara Hanoi. Permainan ini terdiri daripada lima batang tiang, ($N = 5$) dan n cakera. Terdapat dua warna bagi cakera, iaitu biru dan merah. Matlamat Menara Hanoi Dwi-Warna adalah untuk membentuk dua menara cakera berlainan warna di tiang–tiang yang berbeza dengan langkah yang mimima. Bilangan langkah yang diambil untuk membentuk dua menara biru dan menara merah dikaji. Satu persamaan dibentuk untuk $1 \le n \le 3$ cakera. Didapati bahawa, bagi bilangan cakera $n = 2 \; dan \; 3$, formula umumnya adalah $2(n-1)$. Bagi $n = 1$ cakera, hanya satu langkah diperlukan. Graf Hanoi bagi Menara Hanoi Dwi-Warna dilukis dan dibentuk bagi satu, dua dan tiga cakera. Graf Hanoi menunjukkan kesemua langkah yang boleh diambil untuk membentuk dua menara yang berbeza warna, samada langkah yang minima mahupun yang maksima. Graf Hanoi bagi setiap cakera dilukis dengan menggunakan *Virtual Reality Modelling Language (VRML).* Graf Hanoi digambarkan dalam tiga dimensi. Graf Hanoi bagi Menara Hanoi Dwi-Warna berbentuk piramid. Graf Hanoi akan menjadi lebih komplikasi untuk dilukis jika bilangan cakera ditambah. Graf Hanoi menunjukkan repitasi bagi kombinasi piramid-piramid yang dibentuk. Bagi memantapkan lagi corak Graf Hanoi yang diperoleh daripada Menara Hanoi Dwi-Warna, Graf Hanoi bagi empat batang tiang, ($N = 4$) dilukis untuk satu, dua dan tiga cakera. Didapati, repitasi corak tetrahedron dan piramid yang sama diperoleh bagi kedua-dua bilangan tiang $N = 4$ dan $N = 5$.

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LIST OF SYMBOLS

$+$	Addition sign
$-$	Subtraction sign
$*$	Multiplication sign
$/$	Division sum
$=$	Equals to
$<$	Lesser than
$>$	Greater than
$\leq$	Less than or equals to
$\geq$	Greater than or equals to
$\therefore$	Therefore
$\wedge$	To the power of
$perm(...)$	Permanent of
$\|...\|$	Modulus of
$\subseteq$	Subset of
$\in$	Element of
Π	Pi
$\sum$	Summation sign
$\{...\}$	Set of
(a_{ij})	Matrix coordination
o	Omicron

CHAPTER 1

INTRODUCTION

1.1 BACKGROUND OF THE GAME

The Tower of Hanoi game is rather interesting to play. It was invented by E. Lucas in 1883. This game consist a stack of *n* discs arranged from the largest on the bottom to smallest on top placed on a peg, together with two empty pegs. The Tower of Hanoi game asks for the minimum number of moves required to move the stack from one peg to another, where moves are allowed only if they place smaller discs on top of the discs. One has to put his entire mind to play this game. A lot of thinking has to be done as well. This game is also extremely popular among mathematician as it involves mathematics to solve it. Hence, there have been many modifications according to the original version in the market today. The Tower of Hanoi game may look easy to play but do not under estimate this game. When the number of discs increases, it will be harder to solve this game (Weisstein, 2006c).

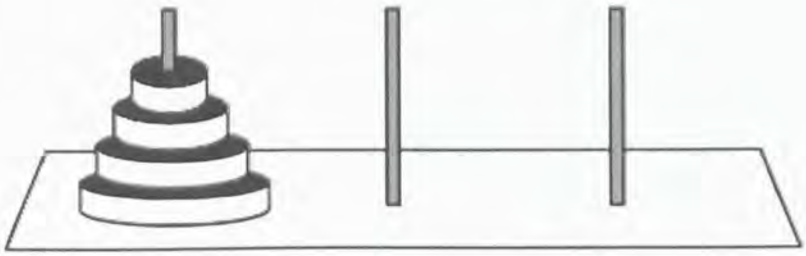

Figure 1.1 The original Tower of Hanoi with 3 pegs and $n = 4$ discs

From the original version of The Hanoi game, many new versions have been invented. This research concentrates on the two colours (red & blue) and consists of $N = 5$ pegs. The discs are arranged from the smallest disc at the top of the tower while the largest disc is placed at the bottom. Red and blue colour discs are arranged alternately. This new variant is known as the "Bi-Colour Pentad Tower". 'Bi-Colour' is due to the two coloured discs (red and blue) that this game carries and also the final tower will be formed such that two stacks of red and blue discs lie together on different pegs, while 'Pentad' simply because of $N = 5$ pegs.

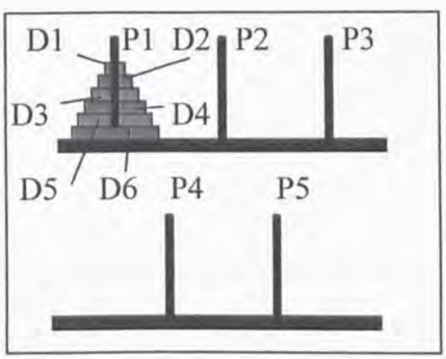

Figure 1.2 Bi-Colour Pentad Tower with $n = 6$ discs

The objective of the Bi-Colour Pentad Tower is to make two different stacks of red and blue tower on different pegs, expect for the first peg. The number of moves is taken on account. Furthermore, the optimal solution of discs moved, which leads to a general mathematical formula will be the prime objective of this game.

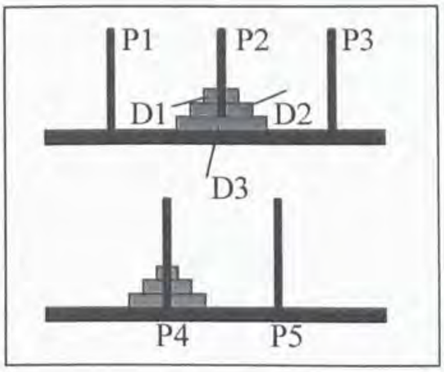

Figure 1.3 The goal of the game is to obtain two different colour towers on different pegs, expect on the first peg, P1

1.2 THE INVENTOR

Francois Eduoard Anatole Lucas was born in April 4th, 1842 in Ameins, France. Lucas was the son of a labourer. He was educated at Ecole Normale in Ameins. He graduated in 1864 from the latter as *Agrege des science mathematiques*. He served as an artillery officer in the Franco-Prussian war and then become professor of mathematics in Paris (Koshy, 2001).

Lucas loved computing and developed plans for a computer, but it never became true. Lucas was famous for his contribution to number theory. He is also known for his four-volume classic on recreational mathematics. Lucas wrote papers on astronomy, weaving, geometry, analysis, combinatorial, mathematical recreations and calculating devices. Lucas also contributed to factoring and primality testing. It was a brief and incomplete study that he made his most important contribution to mathematics. One of his best works was the creation of The Tower of Hanoi game (Williams, 1998).

1.3 APPLICATION OF THE HANOI GAME

The Tower of Hanoi is frequently used in psychological research on problem solving. There also exists a variant of this task called Tower of London for neuropsychological diagnosis and treatment of executive functions.

The Tower of Hanoi is also used as backup rotation scheme when performing computer data backups where multiple tapes or media are involved. As mentioned above, the Tower of Hanoi is popular for teaching recursive algorithms to beginning programming students (Stockmeyer, 2005). Below are some more specific applications of this game:

1.3.1 In Popular Fiction

A popular classic fiction story 'Now Inhale' was created using the concept of the Tower of Hanoi game. In this game, there is a human hero who is a prisoner on a planet where the local custom is to make the prisoner play a game until it is won or lost, and then execution is immediate. The hero is told the game can be one of his own species, as long as it can be played in his cell with simple equipment strictly according to rules which are written down before play and cannot change after play starts, and it has a finite endpoint.

The game and execution are televised planet-wide, and watching the desperate prisoner try to spin the game out as long as possible. It was very popular entertainment for the aliens. The hero knows that a rescue ship might take a year or more to arrive. Instead of being killed the hero chooses to play Towers of Hanoi with 64 disks until his rescue ship arrives. When the aliens realize that a rescue ship has arrived, they were angry, but under their own rules there is nothing they can do about it but to let their prisoners go (Wikipedia, 2006).

Another popular serial fiction, the 'Doctor Who' also uses the Tower of Hanoi Game concept. In one of its series "The Celestrial Toymaker", the 'Toymaker' challenges the Doctor to complete the "Trilogic Game". This Game is actually the Hanoi game with eight rings. It can be solved with the optimal solution in exactly $2^{10}-1$ moves, which sums up to 1023 moves (Wikipedia, 2006).

1.3.2 In Video Games

The Tower of Hanoi game is regularly used in adventure and puzzle games. The game characteristic which is easy to implement and easily recognized makes it well suited to use as a puzzle in a larger graphical game.

Some implementations use straight rings, but others implement it in some other form. The follows is a partial list of games which use the puzzle (Wikipedia, 2006):

(i) Black & White

(ii) Zork Zero

(iii) The Island of Dr. Brain

(iv) The Secret Island of Dr. Quandary

(v) Star Wars; Knights of The Old Republic

1.3.3 In Computers

Tower of Hanoi is also used in the computers backup rotation scheme. A backup rotation scheme is a method for effectively backing up data where multiple media formats are used in the backup process. A common example is the rotation of backup tapes on a regular basis (Wikipedia, 2006).

1.3.4 In Finite State Machines

Tower of Hanoi is also used in the automata machine. An automata machine can be programmed to play the tower of Hanoi Game. By using graphical methods, we can keep track of previous and future moves. Search trees can be used to transform the initial configuration into desired configuration (Dougherty & Giardina, 1988).

1.3.5 In Psychological Research

Often The Tower of Hanoi game also is used in medical field. The most popular usage is for both working memory and inhibition processes. Usually patients with amnesia or suffering from psychological illness will be the subject of study. These patients are given the Tower of Hanoi Game and are required to move one tower to another peg.

These patients' brain and thinking capabilities are measured from the steps they perform to complete their task. These patients' steps are recorded as well. Moreover, they are given to play the game for many times. Each time they play this game the numbers of moves are studied. Hence, we can know how these patients' brain capability by measuring if they can remember the earlier steps and find the optimal moves each time they replay the game (Welsh *et al.*, 1999).

1.3.6 In Teaching

The Tower of Hanoi game is used in teaching students in understanding and finding the code to solve this game. This game is modified to software, known as 'Math Blaster Mystery Math'. This software is designed to foster the development of strategies and analysis in both logic and problem solving skills. The package has four separate activities:

(i) Follow the step, which teaches the steps for solving word problem by breaking the problem into workable parts.
(ii) Weigh the Evidence, which challenges students to stack numbers on scale.
(iii) Decipher the code, which challenges students to make inferences based on their knowledge of mathematics facts.
(iv) Search for a clue, which encourages students to develop definitions and terminology for numbers.

'Math Blaster Mystery Math' covers whole numbers, decimal, percents, fraction, positive and negative numbers, interest and pre-algebra concepts. It extends students' thinking capability (Corcoran, 1989).

1.4 OBJECTIVES OF RESEARCH

The objective of this research is to propose a new variant of The Hanoi Game from the original version which consists of only three pegs and n discs. The colour of these discs and the pegs will be touched.

Two colours (Red and Blue) would be introduced to this game. Instead of just three pegs, five pegs $(N=5)$ will be introduced due to the number of discs that will be used. Five pegs are introduced as there will be three discs. With the colour variant as addition, five pegs will be a simpler feature to solve the game than just three pegs. All together three discs $(1 \leq n \leq 3)$ will be used in this variant. This leads about how to:

(i) Achieve a solution to the Tower of Hanoi problem when the discs are rearranged and coloured

There are no coloured discs in the original Tower of Hanoi game. The Bi-Colour Pentad game has two coloured discs that are arranged alternately with a red colour disc on the top of the tower followed by a blue colour disc at the bottom. A solution of this new variant game needs to be found. The given arrangement of discs is transferred to two towers of the same colours on different pegs.

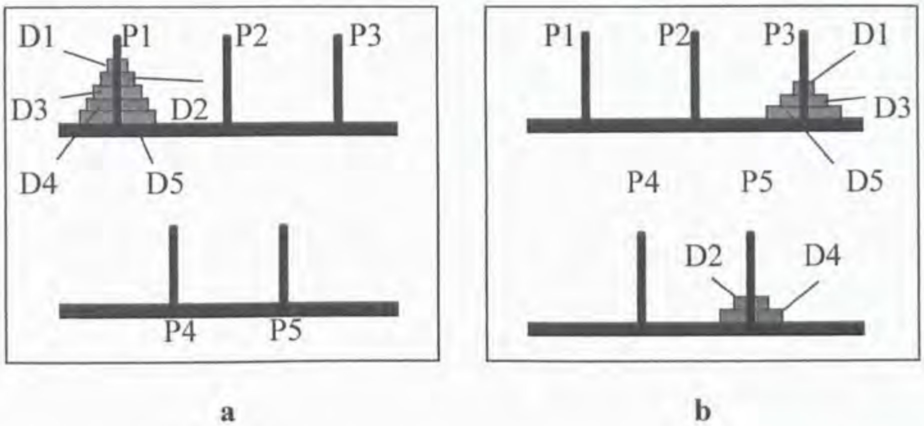

Figure 1.4 **(a)** The discs and pegs arrangements of the Bi-Colour Pentad game with 5 pegs

(b) The objective is to gain 2 different colour towers on different pegs

(ii) Achieve the minimal number of moves to solve the modified Tower of Hanoi game

When playing the Tower of Hanoi game, every move of the discs is extremely important. One different move will cause increment in the number of moves to achieve the goal. Since the aim is to get the minimal number of moves, the movement of the discs are carefully studied. Furthermore, the minimal number of move will lead towards gaining the mathematical formula.

(iii) Study the best and the fastest move (the optimal solution) of the discs in achieving two different towers of red and blue respectively

The optimal solution of the Bi-Colour Pentad game is important to find as it will lead to find the general formula of this game. The Hanoi Graphs provides the entire possible moves for each disc. Thus, the Hanoi Graphs for each disc will be drawn and the optimal move can be found. To compare the Hanoi Graph for $N = 5$ pegs, new graphs are generated for $N = 4$ pegs. By then, the Hanoi Graphs sequence can be studied.

(iv) Achieve a new mathematical formula by studying the pattern of proposed game

A general formula is to be achieved by finding the fastest and the most suitable moves of this game. Here, the pattern is referred on the two towers that are achieved finally. The pattern of discs movement is studied and a formula is created based on the sequence on those numbers.

1.5 SCOPE OF RESEARCH

There are many variants of The Tower of Hanoi Game. This research only touches two colours (Red and Blue) and consists of five pegs $(N = 5)$. The number of discs considered, n are $1 \leq n \leq 3$. The original Tower of Hanoi Game consists of $1 \leq n \leq 64$ discs. Assuming each disc is moved at a rate of one per second with the optimal moves, it would take approximately $2^{64} - 1$ moves. This is exactly 18,446,744,073,709,551,615 moves to complete playing, which is roughly 585 billion years. Since the number of moves is extremely enormous, three discs are chosen as the optimal moves are relevant to be carried out manually. Furthermore, even if more discs are chosen, the Hanoi Graphs will be extremely complicated to be visualised and drawn. Using Virtual Reality Modelling Language (VRML), the Hanoi Graph for three discs it self takes huge space to be carried out. Hence, three discs are chosen as a limit for this research.

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