## MERSENNE PRIMES

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## DECLARATION

I hereby declare that this dissertation contains my original research work. Sources of findings reviewed herein have been duly acknowledged.

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#### Abstract

This dissertation focuses on Mersenne primes, where Mersenne primes are used to generate the corresponding even perfect numbers. Trial-division and Lucas-Lehmer test are used to verify Mersenne primes. The ones digit and tens digit of Mersenne primes are proved by using modular arithmetic. Some patterns of Mersenne numbers are introduced and diagonals of Mersenne's triangle are explored. Even perfect numbers that generated by Mersenne primes are determined by Euclid's theorem and Euler's theorem respectively. The last two digits of even perfect number are proved by using division algorithm and modular arithmetic. Relation of Mersenne primes and even perfect number is studied in triangular number's form.


## NOMBOR PERDANA MERSENNE


#### Abstract

ABSTRAK

Disertasi ini memberi tumpuan terhadap nombor perdana Mersenne, di mana nombor perdana Mersenne dapat digunakan untuk memperolehi nombor sempurna yang genap. Ujian cuba-jaya dan Lucas-Lehmer digunakan untuk menentusahkan nombor perdana Mersenne. Digit terakhir dan kedua terakhir bagi nombor perdana Mersenne dibuktikan dengan menggunakan modulo aritmetik. Corak bagi nombor Mersenne telah diperkenalkan dan pepenjuru bagi segitiga Mersenne telah diselidik. Nombor sempurna yang dijanakan oleh nombor perdana Mersenne dibuktikan dengan teorem Euclid dan teorem Euler masing-masing. Digit kedua terakhir bagi nombor sempurna dibuktikan dengan menggunakan algoritma pembahagian and modulo aritmetik. Hubungan antara nombor perdana Mersenne dan nombor sempurna genap dikaji dalam bentuk nombor segitiga.


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## LIST OF SYMBOLS

| $M_{p}$ | Mersenne number, $2^{p}-1$ |
| :---: | :---: |
| N | set of natural numbers $\{1,2, \ldots\}$ |
| Z | set of integers $\{0, \pm 1, \pm 2, \ldots\}$ |
| $(a, n)$ | $\operatorname{gcd}$ of $a$ and $n$ |
| gcd | greatest common divisor |
| $k \mid m$ | $k$ divides $m$ |
| $\epsilon$ | subset to |
| $\notin$ | is not subset to |
| $S(n)$ | sum of $n$ |
| * | multiplication function |
| $\|G\|$ | the modulus of $G$ |
| $\cup$ | union |
| $\subseteq$ | subset to or equal to |
| $\sigma(n)$ | sum of the divisors of $n$ |
| $\mathrm{Z}_{r}$ | set of congruence classes of mod (r) |
| $I(r)$ | the identity function equal to $\left\lfloor\frac{1}{r}\right\rfloor$ for all r |
| $\rho$ | the reduced form of $\alpha$ modulo $I(r)$ |
| $a^{m} \equiv 1$ | $a^{m}$ is congruent to 1 modulo ( $n$ ), also denoted by |
|  | $a^{m} \equiv 1(\bmod n)$ |

## CHAPTER 1

## INTRODUCTION

### 1.1 Background

Numbers have exercised their fascination since the dawn of civilizations. Each number has its own characteristics and beauties. Prime numbers are the basic building blocks of mathematical world. Every natural number except one is either prime or is made up of primes multiplied together.

In this research, Mersenne numbers which are one of the forms of prime numbers will be studied. An integer $p>1$ is said to be prime if the only positive divisors of $p$ are one and $p$ itself. Mersenne numbers have played an important role in computational number theory. At first, Mersenne investigated prime numbers and he tried to find a formula that would represent all prime numbers. Although he failed in this, he found out that his work on the numbers $\left(2^{p}-1\right)$, where $p$ is prime, can be used to represent large primes.

This study shall be focused on Mersenne primes, which is a special type of prime numbers. A Mersenne prime number is a prime number that happens to be a Mersenne number.

A Mersenne prime will always have a value equal to $\left(2^{p}-1\right)$ where $p$ is one of a selected list of positive prime numbers. Mersenne primes will be tested by using the primality test for Mersenne numbers.

There are many people searching for Mersenne primes but not many people interested in doing research about the characteristics of Mersenne primes. There are many great explorations about Mersenne primes. Therefore, in this study, observations on the characteristics of Mersenne primes such as the patterns of numbers shall be focused. Besides, some patterns of Mersenne numbers shall be studied in this research.

Besides, this study will concentrate on using the Mersenne primes to obtain perfect numbers or more specifically even perfect numbers. Odd perfect numbers are different matters. It is not known whether the odd perfect numbers exist or not. Mathematicians have not been able to prove that none exist so far. Moreover, the patterns of even perfect numbers generated by Mersenne primes shall be studied.

Before going deeper in this study, it is important to review some historical facts about Mersenne numbers. The objectives of this study will be highlighted as the goal for this research.

### 1.2 Biography of Marin Mersenne

Marin Mersenne was a $17^{\text {th }}$ century mathematician, who studied the Mersenne numbers. He was born on September 8, 1588 in Maine, France. Mersenne received education at the Colleges of Mans and the Jesuit College of La Fleche. He studied theology at Sorbonne University for two years. After that, he taught philosophy at Minim Convent in Nevers between 1614 and 1618. In 1619, he returned to Paris and remained in Minims de I'Annociade near Palace Royale for the rest of his life (Rosen, 2000).

Father Marin Mersenne had been gathered the scientists, mathematicians and philosophers to discuss about their respective discoveries or ideas. Mersenne had many meetings in his cell in Minim Convent with Pascal, Fermat and other unknown mathematicians from 1635 until his death in 1648. After Mersenne's death, they continued their discussions at private houses in Paris, including Pascal's.

From 1625 onwards, Mersenne made his efforts in bringing mathematical and scientific information. It was Mersenne who made widely known that the physicist's demonstration of atmospheric pressure through the rising of a column of mercury in a vacuum tube followed his visit to Torricelli in Italy in 1645.

Marin Mersenne also became the main channel of communications between Fermat, Frenicle and Descartes. They kept exchanging letters and had determined the kind of problems they chose to investigate. In a letter written in 1643, Mersenne
requested Fermat to found out the factors of number $100,895,598,169$. Fermat had solved the problem and the two primes were 898423 and 112303.

Mersenne had published several books of mathematical sciences including Synopsis Mathematica (1626), Traite de l'Harmonie Universelle (1636-37) and Universae Geometriae Synopsis (1644). He was actively in popularizing Galileo's investigations. In 1634, he bought out a version of Galileo's Discorsi under the title Les Mecaniques de Galilee. A year after its original publication, Mersenne did translations for Galileo's Discorsi - a treatise analyzing projectile motion and gravitational acceleration into French in 1639. Mersenne's greatest contribution to scientific movement was his rejection of the traditional interpretation to natural phenomena (Burton, 2006).

### 1.3 History of Mersenne Numbers

Mersenne conjectured that number of the form $2^{p}-1$ is prime for the value of $p$ that is prime (Brown, 1978). However, in 1536 Hudalrichus Regius found that it fails when $p$ is 11 , the number $2^{11}-1=2047=23 \cdot 89$. It was not a Mersenne prime number as it contained two prime factors, which were 23 and 89 , as stated in his work entitled Utriusque Arithmetices.

By 1588 , Pietro Cataldi had verified that $2^{17}-1$ and $2^{19}-1$ were primes, but he also stated that $2^{p}-1$ was also prime for $23,29,31$ and 37 without giving a proof. In 1640 , Fermat showed Cataldi was wrong about 23 and 37 . Later in 1738, Euler showed Cataldi
was wrong about 29. In 1772, Leonard Euler showed Cataldi's assertion about 31 was correct. He ascertained that $2^{31}-1=2,147,483,647$ is a prime number by using trial division (Burton, 2006).

In 1644, Father Marin Mersenne stated in his Cognitata Physica-Mathematica (without a proof) that the numbers $2^{p}-1$ were prime for $p=2,3,5,7,13,17,19,31,67$, 127 and 257 and were composite for all other positive integers $p<257$ (Rosen, 2000). No one knew how he arrived at this claim.

The Mersenne numbers are composite for the following primes like $257=2^{8}+1$, $1021=2^{10}-3,67=2^{6}+3$ and $8191=2^{13}-1$. Over the years, it had been found that Mersenne was wrong about five of the primes less than or equal to 257 . He stated two primes that did not lead to a prime, 67 and 257 . He also missed three that did, 61,89 , and 107 (Uhler, 1948). It took about three centuries to settle his claim.

In 1876, Edouard Lucas verified that $2^{127}-1$ was a prime number. This number is, written in full, is $170,141,183,460,469,231,731,687,303,715,884,105,727$ and contains 39 digits (Lines, 1986). He doubted this result but it was confirmed by Fauquembergue in 1914. It was the largest prime to be discovered without the aids of modern calculating.

In 1883, Pervushin showed that $2^{61}-1$ was prime. In the early 1900 's Powers found that Mersenne had missed the primes $2^{89}-1$ and $2^{107}-1$ (Ondreika, 1986). By

1947, Mersenne's range, $p<257$, had been completely checked. It was determined that the correct list is given by $p=2,3,5,7,13,17,19,31,61,89,107$ and 127 .

Uhler also contributed in seeking Mersenne numbers. On $27^{\text {th }}$ November of 1947, Uhler finished investigating, by application of the Lucasian sequences four, 14, 194, $37634, \ldots$, the factorizability of Mersenne's number $M_{193}=2^{193}-1=12554203470773$ 361527671578846415332832204710888928069025791 . The $192^{\text {nd }}$ residue had the value 5424570125193908141321143009568020463304970794324280151282 which, being non-zero, shows that $M_{193}$ is composite (Uhler, 1948).

### 1.4 Study Objectives

The main objectives of this study are:

## 1. To Understand Mersenne Numbers

Besides investigating the characteristics of Mersenne primes including its patterns of numbers, some patterns of Mersenne numbers shall be studied.
2. To Verify Mersenne Primes by Using the Primality Tests of Mersenne

## Numbers

This study will focus on using two primality test in testing Mersenne primes, which consists of trial division and Lucas-Lehmer test.

## 3. To Use Mersenne Primes to Generate Perfect Numbers or More Specifically Even Perfect Numbers

Based on this objective, perfect numbers generated by Mersenne primes shall be determined by using Euclid's theorem and Euler's theorem respectively.
4. To See the Patterns of Even Perfect Numbers Generated by Mersenne Primes It is known that perfect numbers end with either six or eight. This study will focus on other patterns of perfect numbers generated by Mersenne primes.

### 1.5 Study Scope

There are many ways in determining prime numbers. However, this research only focuses in studying Mersenne numbers, especially where the Mersenne primes are used to generate even perfect numbers.

## CHAPTER 2

## LITERATURE REVIEW

### 2.1 Introduction

In this section, the process of people searching for Mersenne primes, which can be said as a centuries-old research shall be focused. It can be divided into three categories: before the advent of electronic computers, after the advent of electronic computers and Great Internet Mersenne Prime Search (GIMPS). From the Table 2.1, it shows the list of Mersenne primes and it will be used to discuss the characteristics of Mersenne primes in the latter chapter. (For Table 2.1, please refer to Appendix A, page 113). Reasons that motivate people in seeking Mersenne primes will be discussed in view of the applications of Mersenne numbers in the areas of computer or computational sciences, mathematics and engineering. Some historical facts about perfect numbers shall be reviewed in order to get know about the relation between Mersenne primes and even perfect numbers.

### 2.2 History in Searching for Mersenne Primes

### 2.2.1 Before the Introduction of Electronic Computers

Before the advent of electronic computers, the search for Mersenne primes was full with calculation errors as it involved large prime numbers. Pietro Cataldi proved that the Mersenne prime generated by $p=17$ and 19 , denoted by $M_{17}=131071$ and $M_{19}=524289$ were primes in 1588 . He made assertions that $M_{p}$ were primes for $p=23,29,31$, and 37 but only $M_{31}$ was correct. In 1644, Marin Mersenne stated (without proving) in his Cognita Physica-Mathematica that $M_{p}$ were primes for $p=2,3,5,7,13,17,19,31,67$, 127 and 257 and were composite for all other positive integers $p<127$ (Rosen, 2000).

In 1772, it was Euler who proved that $M_{31}=2147483647$ was prime. By that time, only eight Mersenne primes had been found (Rosen, 2000). In 1876, with the introduction of the use of electronic hand calculators, Lucas had managed to get the $12^{\text {th }}$ Mersenne prime that was $M_{127}$ which consisted of 39 digits. It was disappointing that none of the Mersenne prime was being found since then. At the same year, Lucas initiated a theory about the test for the primality of Mersenne numbers and Lehmer had used the theory to make it into a simple test in 1930. The test was computed by using the modular of the numbers $\left(2^{p-1}\right)$. Using the test and calculators, several Mersenne primes were added to the list of Mersenne primes (Spencer, 1989).

In 1876, Edouard Lucas stated that $M_{67}$ was composite using the test he had proposed without providing a factorization. After 27 years, an American mathematician Frank Nelson Cole succeeded in finding the factors of $M_{67}$, that is $M_{67}=193,707,721$ $.761,838,257,287$ and he presented this result on a paper with unassuming title "On the Factorization of Large Numbers" at a meeting of the American Mathematical Society in the October of 1903. Between 1876 and 1947, the numbers $M_{61}, M_{89}, M_{107}$ and $M_{127}$ were proved to be primes (Rosen, 2000).

The primality tests for all primes, $p<257$ was done in 1947 with the aid of mechanical calculating machines. The correct list for the Mersenne primes were numbers generated by $p=2,3,5,7,13,17,19,31,6189,107$ and 127 . Mersenne made five mistakes as he included wrongly $p=67$ and 257 into the list and he did not include $p=61$, 89 and 107 ( $p=67$ or 257 results in composites and $p$ equal to 61,89 or 107 results in primes) (Ribenboim, 1996). Only twelve Mersenne primes had been discovered before the advent of modern computers, the last was $M_{107}$ which was found by Powers in 1914.

### 2.2.2 After the Introduction of Electronic Computers

Since the advent of computers, it provides a more efficient way in seeking Mersenne primes. There are many Mersenne primes have been discovered in a fairly steady rate.

In 1952, Raphael Robinson had discovered five Mersenne primes from the $13^{\text {th }}$ $M_{p}$ to $17^{\text {th }} M_{p}$ using SWAC (the National Bureau of Standards Western Automatic Computer) with the help of D.H and Emma Lehmer. The Mersenne primes were $M_{521}$, $M_{607}, M_{1279}, M_{2203}$ and $M_{2281}$. The thirteenth and fourteenth Mersenne prime numbers were found in $30^{\text {th }}$ January, which was the first day Raphael Robinson ran the electronic computer (SWAC) using the Lucas-Lehmer test while the other three were discovered in the following nine months (Rosen, 2000).

In 1957, Riesel discovered the $18^{\text {th }} M_{p}$ with the help of Swedish's first electronic computer, the BESK (Binary Electronic Sequence Calculator). Hurwitz found the $19^{\text {th }}$ $M_{p}$ and $20^{\text {th }} M_{p}$ which consisted of 350484991 and 608580607 decimal digits respectively in 1961 using the IBM 7090. In 1963, it was Donald B. Gillies of the University of Illinois who discovered the $21^{\text {st }} M_{p}, 22^{\text {nd }} M_{p}$ and $23^{\text {rd }} M_{p}$ using the ILLIAC 2. The twenty-third Mersenne prime was being advertised on the university's postage meter as " $2{ }^{11213}-1$ is prime" (Koshy, 2002).

Bryant Tuckerman of International Business Machines (IBM) found the $24^{\text {th }}$ Mersenne prime, $2^{19937}-1$, with the help of an IBM System 360/91. It has 6,002 digits, which begins and ends $4315424797 \ldots 0968041471$ (Tuckerman, 1971).

In 1978, two high school students Laura Nickel and Landon Noll discovered the $25^{\text {th }} M_{p}$ using a Control Data Cyber 174 computer at California State University,

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