NUMERICAL PERFORMANCE OF A FAMILY OF
PRECONDITIONED GAUSS-SEIDEL METHODS
FOR ONE AND TWO ASSET STANDARD
OPTION PRICINGS

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14 May 2012
ABSTRACT

NUMERICAL PERFORMANCE OF A FAMILY OF PRECONDITIONED GAUSS-SEIDEL METHODS FOR ONE AND TWO ASSET STANDARD OPTION PRICINGS

Development in numerical techniques has greatly influenced the advancement of quantitative finance in solving any mathematical models concerned efficiently. Recently, solving the Black-Scholes partial differential equations (PDEs), the option pricing models have attracted many mathematicians to contribute and enhance the existing analytical and numerical solutions. Option is a financial instrument which gives its holder the right without obligation, to trade a certain asset in future at a stated price. The most traded option styles in the market are European and American options. In this thesis, the scope of study covers the pricing of European and American options underlying one- and two-asset which are modelled by one- and two-dimensional Black-Scholes PDEs respectively. Full-, half-, and quarter-sweep Crank-Nicolson finite difference approximations are used to discretize the Black-Scholes PDEs. Hence, linear systems made up of three- and nine-point stencils for one- and two-dimensional problems respectively are generated from the corresponding approximation equations. To solve the European options pricing, a family of preconditioned Gauss-Seidel (GS) methods which consists of Full-Sweep Gauss-Seidel (FSGS), Half-Sweep Gauss-Seidel (HSGS), Quarter-Sweep Gauss-Seidel (QSGS), Full-Sweep Modified Gauss-Seidel (FSMGS), Half-Sweep Modified Gauss-Seidel (HSMGS), Quarter-Sweep Modified Gauss-Seidel (QSMGS), Full-Sweep Improving Modified Gauss-Seidel (FSIMGS), Half-Sweep Improving Modified Gauss-Seidel (HSIMGS) and Quarter-Sweep Improving Modified Gauss-Seidel (QSIMGS) iterative methods are proposed. Due to early exercise of American options, linear complementarity problems (LCPs) are formed and a family of projected preconditioned GS methods are developed. Several numerical experiments for the families of preconditioned GS and projected preconditioned GS methods are also implemented. The performances of these iterative methods are analyzed by observing the number of iterations, computational time and root mean squared error (RMSE). Based on the results for all the problems, QSIMGS and Quarter-Sweep Projected Improving Modified Gauss-Seidel (QSPIMGS) iterative methods yield the fastest number of iterations and computational time among the tested methods. Moreover, the accuracies of QSIMGS and QSPIMGS methods are in good agreement with GS and projected GS (PGS) methods respectively. Overall, it can be concluded that QSIMGS and QSPIMGS iterative methods are very efficient in terms of number of iterations and computational time in solving option pricing models.
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<td>Adaptive Gauss-Seidel</td>
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<td>AM</td>
<td>Arithmetic Mean</td>
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<td>AOR</td>
<td>Accelerated OverRelaxation</td>
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<td>BICGSTAB</td>
<td>Biconjugate Gradient Stabilized</td>
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<td>CBOE</td>
<td>The Chicago Board of Exchange</td>
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<td>CN</td>
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<td>EDG</td>
<td>Explicit Decoupled Group</td>
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<td>EG</td>
<td>Explicit Group</td>
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<td>LCP</td>
<td>Linear Complementarity Problem</td>
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<td>LOD</td>
<td>Locally One-Dimensional splitting</td>
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<td>MEDG</td>
<td>Modified Explicit Decoupled Group</td>
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<td>MEG</td>
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<td>MEGSOR</td>
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<td><strong>RMSE</strong></td>
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<td>Successive Over-Relaxation</td>
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LIST OF SYMBOLS

\( \alpha \) Parameter of IMGS iterative method

\( \beta \) Parameter of Modified Adaptive Gauss-Seidel method

\( \partial \) Derivative

\( \epsilon \) Error tolerance

\( \epsilon \) A small positive number to avoid dividing by a too small number


\( \mu \) Mean rate of return

\( \mathcal{N}(.) \) Cumulative normal distribution

\( \Pi \) Portfolio value

\( \theta \) Theta scheme, a general scheme corresponds to explicit, implicit and CN schemes

\( \rho \) Correlation

\( \sigma \) Volatility of the underlying asset

\( \omega \) Parameter of SOR iterative method

\( \text{A} \) Coefficient matrix of a linear system

\( \text{B} \) Standard Brownian motion

\( b \) Cost of carry rate (risk free interest rate minus dividend)

\( \text{C} \) A preconditioned matrix proposed by Milaszewicz (1987)

\( c \) Unchanged movement of the underlying asset in a trinomial tree

\( D \) Dividend

\( d \) Downward movement of the underlying asset in a tree

\( g \) Payoff function

\( h \) Step size

\( \text{I} \) Identity matrix
K  Strike price
L  Strictly lower triangular matrix
M  A non-singular matrix from the splitting of A
m  Time step
max(.) Return the greatest value among the values in the function.
N  A matrix from the splitting of A
n  Mesh size
P  Preconditioner
p  Probability
q  General notation corresponds to full-, half- and quarter-sweep approaches
R  A preconditioning matrix proposed by Morimoto et al. (2003)
r  Risk free interest rate
S  A preconditioning matrix of MGS and IMGS iterative methods
Sm A preconditioning matrix proposed by Kotakemori et al. (2002)
s  Price of an underlying asset
s_{\text{max}}  Maximum asset price
s(t) Price of an underlying asset at time t
T  Maturity time
t  Time
\Delta t  Step size of time
U  Strictly upper triangular matrix
u  Upward movement of the underlying asset in a tree
v  Option's value
W(.) Cumulative bivariate normal distribution
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>w</td>
<td>Fixed quantities of the underlying asset</td>
</tr>
<tr>
<td>x</td>
<td>Value for $s_1$</td>
</tr>
<tr>
<td>y</td>
<td>Value for $s_2$</td>
</tr>
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</table>
CHAPTER 1

PRELIMINARY

1.1 Background
Numerical methods are techniques which formulate and solve mathematical problems, mostly involving a large number of tedious arithmetic operations (Chapra and Canale, 2006). In fact, numerical methods play a vital role in problem solving owing to the rapid progress of high speed, storage, processing and sophisticated computer technology. Generally, problems in engineering and computational science are widely dependent on numerical methods, for example computational chemistry, fluid dynamics, astronomical structures and structural analysis.

Nevertheless, numerical methods are starting to be established in the area of finance. Various problems in quantitative finance can be analyzed via numerical methods including, but not limited to derivative pricing, portfolio selection and financial econometrics (Brandimarte, 2006; Fusai and Roncoroni, 2008). The earliest study recorded on computational finance was in 1977 where Brennan and Schwartz (1977) grounded on the finite difference method in pricing options. In fact, the most discussed problem in the field of computational finance is the option pricing. In 1995, the first international conference on computational finance was conducted at Stanford University and shortly thereafter was the birth of The Journal of Computational Finance (Tavella, 2002). These implied the popularity and achievement of computational finance whereby a mass of research can be done and is ongoing.

However, the mathematical modelling of option pricing was initiated much earlier by French mathematician, Bachelier (1900) in his thesis which applied the Brownian motion concept to price the option. He was the first who quantified the financial problem with mathematical and physics theories. In spite of this, for many years not much work was heard regarding derivatives modelling. It was known
that Bachelier’s work was not a great achievement during his era, plus Poincaré, his supervisor did not appreciate his work much (Merton 1995; Wilmott, 2002; Almgren 2002). Fortunately, after more than half a century, Bachelier’s thesis was rediscovered by Samuelson who laid the foundation of later option pricing theories (Merton, 1995; Wilmott, 2002). Then, Thorp and Kassouf (1967) came into the picture and managed to develop a stock option pricing formula as he did in warrant pricing via fitting a curve to actual warrant prices. Subsequently, Black and Scholes (1973) and Merton (1973) improved the idea of Thorp and Kassouf (1967) and presented the well known Black-Scholes model to price option successfully. This brought the end of the subject and eventually listed options started to trade in The Chicago Board of Exchange (CBOE) for the first time in 1973 (Merton, 1995).

According to Howison (1995) in a Royal Society of London discussion meeting on “Mathematical Models in Finance”, he pointed out in future, more and more exotic options would be in the market but the modelling of them could be built on the basis of Black-Scholes model and would not cause much problem. However, the challenge is in formulating efficient and fast numerical computation for derivative security values involving a very large number of options (Howison, 1995). So, motivated by the challenge from Howison (1995), this thesis is exploring an efficient and fast numerical computation algorithm for solving options pricing.

1.1.1 Introduction to Options
Options can be defined as a kind of derivatives which gives the holder the right, without obligation to trade a certain underlying asset in future with the strike price. The strike price is the price of the asset prescribed in the contract. In other way of saying, an option gives the holder the privilege to do something, but the holder can choose not to exercise this right (Hull, 2009). There are two types of options. A call option offers the right to buy the underlying asset while a put option gives the right to sell the underlying asset. Normally, the underlying asset could be a stock, foreign currency, commodity, index, properties and the like. The asset price stated in the option is known as strike or exercise price and the end life of an option is called the expiration or maturity date.
The reasons the investors acquire an option are for hedging, speculation or arbitrage (Hull, 2009). Hedging protects the investors against unfavourable underlying asset price movements in future while still can benefit from the desired price movements by not exercising the option. On the other hand, some investors like to speculate and take a position in the market. They are actually betting for the price to rise or drop in order to gain profit. The advantage is regardless how bad the situation is, the speculator only losses the amount paid for the option. The third group is the arbitrageur who benefit from riskless profit by simultaneously trading in two or more markets. However, this arbitrage circumstances seldom occurs (Hull, 2009).

1.1.2 Styles of Options
In the finance world, there are many styles of options. Basically, the standard well-defined and widely traded ones are the European and American styles of options which have nothing to do with geographical location. They are also termed as the plain vanilla options. A European option can be exercised only at maturity whereas an American option can be exercised at any time up to maturity (Hull, 2009). Normally, European options are traded over-the-counter while American options are traded on future exchanges. Most index options are European whereas equities and stock are represented by American options.

Besides these are the exotic options which were designed by financial engineers for a number of reasons. Some example are the nonstandard American option, forward start option, compound option, chooser option, barrier option, binary option, lookback option, shout option, Asian option, Russian option, executive option, power option, log option, fade-in option, ratchet option, time-switch option, option on option, mirror option, option involving several assets and many others (Hull, 2009; Haug, 2007).

The most famous exotic options are the path dependent options in which the payoff structures are depending on the underlying asset prices path history during the whole or part of the life of the option (Kwok, 2008). For instance, they are the barrier option, the lookback option, the Asian option and the Russian
option. Barrier options can be nullified, activated or exercised when the asset price reaches a certain level during the life of the option (Hull, 2009; Kwok, 2008). The lookback option is an option in which the payoff depends on the maximum or minimum asset price achieved during the life of the option. An average price of the underlying asset price over particular period of the life of the option is the payoff of an Asian option. The Russian option is also known as the perpetual American lookback option. It gives the holder the historical maximum value of the asset price upon exercising the option and the option has no predetermined expiration date (Kwok, 2008).

1.2 Basic Mathematical Concepts in Problems Solving

Many mathematical models can be expressed in either ordinary differential equations (ODEs) or partial differential equations (PDEs). This study focuses on the PDEs, and a brief introduction on PDEs will be given. Then, the Taylor series expansion is elaborated as it is a foundation in approximation derivation of PDE. As discretization of PDE will generate a linear system, hence various algebraic matrices are presented as well.

1.2.1 Partial Differential Equations

PDE is an equation containing partial derivatives of an unknown function of more than two independent variables. A PDE is considered to be linear if the unknown function and all its derivatives are linear, with coefficients depending only on the independent variables. The order of a PDE is determined by the highest order of partial derivatives in the PDE (Chapra and Canale, 2006).

Most problems are modelled by second order linear PDEs which can be expressed in a general form

\[ a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + d = 0, \]

where \( a, b, c \) and \( d \) are constants or functions of independent variables \( x \) and \( y \) and \( u \) is a dependent variable.
REFERENCES


